Math 252 Homework # 2
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1. Let $R = \mathbb{Z}[\sqrt{-n}] = \{a + b\sqrt{-n} \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$, where n is any integer greater ≥ 2. Let

$$N : R \longrightarrow \mathbb{Z}^{\geq 0} \qquad \text{by} \qquad N(a + b\sqrt{-n}) = a^2 + b^2 n.$$

Note that N is a norm on R (called the "field norm"). Check for yourself that $N(\alpha\beta) = N(\alpha)N(\beta)$, for all $\alpha, \beta \in R$ (do not hand in this exercise; but do hand in the following ones).

(a) Prove that ± 1 are the only units in R.

(b) Prove that if p is a positive prime in $\mathbb{Z}$ with $p < n$, then p is irreducible in R.
Prove also that $\sqrt{-n}$ is irreducible in R.
[Hint: Consider all cases at once by contradiction—apply N to a factorization.]

(c) Prove that that $R/(\sqrt{-n}) \cong \mathbb{Z}/n\mathbb{Z}$.
[Hint: Let $\phi : R \to \mathbb{Z}/n\mathbb{Z}$ by $\phi(a + b\sqrt{-n}) = a(\text{mod } n)$. Show that ϕ is a surjective ring homomorphism with $\ker \phi = (\sqrt{-n})$.]

(d) Deduce from (c) that if n is not a prime, then R is not a U.F.D.

(e) y: Show if n is an odd prime then R is not a UFD as well.
[Hint: Use the methods above with $1 + \sqrt{-n}$ in place of $\sqrt{-n}$.]

2. Prove that the quotient ring $\mathbb{Z}[i]/(1 + i)$ is isomorphic to the field $\mathbb{Z}/2\mathbb{Z}$.

3. Prove that the ideals (x) and (x, y) are prime ideals in the polynomial ring $\mathbb{Q}[x, y]$ but only the latter ideal is maximal.

4. Exhibit *all* the ideals in the quotient ring $F[x]/(p(x))$, where F is a field and $p(x)$ is a polynomial of degree ≥ 1 in the polynomial ring $F[x]$ (describe them in terms of the factorization of $p(x)$).
[Hint: Suppose the nonconstant polynomial $p(x)$ factors (uniquely) as

$$p(x) = c q_1(x)^{e_1} q_2(x)^{e_2} \cdots q_r(x)^{e_r}$$

where $c \in F^{\times}$, $q_1, q_2, \ldots, q_r$ are distinct irreducibles in $F[x]$ and $e_i \in \mathbb{Z}^{+}$. Use Lattice Isomorphism Theorem for rings and the fact that $F[x]$ is a P.I.D.]