Math 252 Homework # 3
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1. Prove that the following polynomials are irreducible in $\mathbb{Z}[x]$ (briefly justify answers):

   (a) $x^4 - 4x^3 + 6$,

   (b) $x^6 + 30x^5 - 15x^3 + 6x - 120$,

   (c) $x^4 + 4x^3 + 6x^2 + 2x + 1$ [Hint: Substitute $x - 1$ for x.],

   (d) $\dfrac{(x+2)^p - 2^p}{x}$, where p is an odd prime.

2. Prove that $x^3 + nx + 2$ is irreducible in $\mathbb{Z}[x]$ for all integers $n \neq 1, -3, -5$.

3. Factor each of the two polynomials: $x^8 - 1$ and $x^6 - 1$ into irreducibles over each of the following rings: (a) $\mathbb{Z}$, (b) $\mathbb{Z}/2\mathbb{Z}$, (c) $\mathbb{Z}/3\mathbb{Z}$.

Let F be any field and let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in F[x]$. The *derivative*, $D_x(f(x))$, of $f(x)$ is defined by

$$D_x(f(x)) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$$

where, as usual, $na = a + a + \cdots + a$ (n times) in the field F. Note that $D_x(f(x))$ is again a polynomial with coefficients in F. For example, if $f(x) = x^4 + x^3 + x^2 + x + 1 \in \mathbb{F}_2[x]$, then $D(f(x)) = x^2 + 1$, because the terms $4x^3$ and $2x$ are zero in $\mathbb{F}_2[x]$.

The polynomial $f(x)$ is said to have a *multiple root* if there is some field E containing F and some $\alpha \in E$ such that $(x-\alpha)^2$ divides $f(x)$ in $E[x]$. For example, the polynomial $f(x) = (x-1)^2(x-2) \in \mathbb{Q}[x]$ has $\alpha = 1$ as a multiple root and the polynomial $f(x) = x^4 + 2x^2 + 1 = (x^2+1)^2 \in \mathbb{R}[x]$ has $\alpha = \pm i \in \mathbb{C}$ as multiple roots. We shall prove in Section 13.5 that a nonconstant polynomial $f(x)$ has a multiple root if and only if $f(x)$ is *not* relatively prime to its derivative (which can be detected by the Euclidean Algorithm in $F[x]$).

4. Use the derivative criterion described above to determine whether the following polynomials have multiple roots (*do not factor these polynomials*):

   (a) $x^3 - 3x - 2 \in \mathbb{Q}[x]$

   (b) $x^3 + 3x + 2 \in \mathbb{Q}[x]$

   (c) $x^6 - 4x^4 + 6x^3 + 4x^2 - 12x + 9 \in \mathbb{Q}[x]$

   (d) Show for any prime p and any $a \in \mathbb{F}_p$ that the polynomial $x^p - a$ has a multiple root.

5. Show that the polynomial $(x-1)(x-2)\cdots(x-n) - 1$ is irreducible over $\mathbb{Z}$ for all $n \geq 1$. [Hint: If the polynomial factors, consider the values of the factors at $x = 1, 2, \ldots, n$.]