Math 252 Homework # 4
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Throughout these exercises R is a ring with 1 (not necessarily commutative).

1. An element m of the R-module M is called a torsion element if $rm = 0$ for some *nonzero* element $r \in R$. The set of torsion elements is denoted

$$\mathrm{Tor}(M) = \{m \in M \mid rm = 0 \text{ for a nonzero } r \in R\}.$$

   (a) Prove that if R is an integral domain then $\mathrm{Tor}(M)$ is a submodule of M (called the *torsion submodule* of M).

   (b) Give an example of a ring R and an R-module M such that $\mathrm{Tor}(M)$ is not a submodule. [Consider the torsion elements in the R-module $M = R$ for suitable ring R.]

2. If N is a submodule of the R-module M, the *annihilator* of N in R is defined to be

$$\mathrm{Ann}(N) = \{r \in R \mid rn = 0 \text{ for all } n \in N\}.$$

   Prove that the annihilator of N in R is a 2-sided ideal of R. (Do not assume R is commutative.)

3. Let F be any field, let $V = F^n$ and let $(a_1, a_2, \ldots, a_n)$ be a fixed vector in V. Prove that the collection of elements $(x_1, x_2, \ldots, x_n)$ of V with

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

   is a subspace of V, and determine the dimension of this subspace. [Hint: Show it is a kernel.]

4. Let V be a vector space of dimension n over any field F. Let ϕ be a linear transformation from V to itself that satisfies $\phi^2 = 0$. Prove that the image of ϕ is contained in $\ker\phi$. Deduce from this that the rank of ϕ is at most $n/2$.

5. Let V be a vector space of dimension n over any field F. Let ϕ be a linear transformation from V to itself that satisfies $\phi^2 = \phi$.

   (a) Prove that image $\phi \cap \ker\phi = 0$.

   (b) Prove that $V = \text{image } \phi \oplus \ker\phi$.

   [A linear transformation ϕ satisfying $\phi^2 = \phi$ is called an *idempotent* linear transformation. This exercise proves that idempotent linear transformations are simply projections onto some subspace.]