Math 252 Homework # 5

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1. (a) Show that $p(x) = x^3 + 9x + 6$ is irreducible in $\mathbb{Q}[x]$.

   (b) Let θ be a root of $p(x)$. Find the (multiplicative) inverse of $1+\theta$ in $\mathbb{Q}(\theta)$.

2. (a) Show that $q(x) = x^3 - 2x - 2$ is irreducible in $\mathbb{Q}[x]$.

   (b) Let θ be a root of $q(x)$. Compute each of the following in $\mathbb{Q}(\theta)$ (simplify answers):

      (i) $(1+\theta)(1+\theta+\theta^2)$, and

      (ii) $\dfrac{1+\theta}{1+\theta+\theta^2}$.

3. Determine with justification the degree over $\mathbb{Q}$ of the following:

   (a) $2+\sqrt{3}$, and

   (b) $1+\sqrt[3]{2}+\sqrt[3]{4}$.

4. Let $F = \mathbb{Q}(i)$. Prove that the following are both irreducible in $F[x]$:

   (a) $x^3 - 2$, and

   (b) $x^3 - 3$.

5. Determine the degree of the extension $\mathbb{Q}(\sqrt{3+2\sqrt{2}}\,)$ over $\mathbb{Q}$.

6. Let $\sqrt{3+4i}$ denote the square root of the complex number $3+4i$ that lies in the first quadrant and let $\sqrt{3-4i}$ denote the square root of $3-4i$ that lies in the fourth quadrant. Prove that $[\mathbb{Q}(\sqrt{3+4i}+\sqrt{3-4i}\,):\mathbb{Q}] = 1$.

7. Prove that if F is any field and $[F(\alpha):F]$ is odd then $F(\alpha) = F(\alpha^2)$.

8. Let K/F be an algebraic extension and let R be a *ring* contained in K and containing F. Prove that R is a field.