Math 252 Homework # 6

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1. Determine the splitting field and its degree over $\mathbb{Q}$ for $x^4 + 2$. (Justify your answer.)

2. Determine the splitting field and its degree over $\mathbb{Q}$ for $x^4 + x^2 + 1$. (Justify your answer.)

3. Determine the splitting field and its degree over $\mathbb{Q}$ for $x^6 - 4$. (Justify your answer.)

4. Suppose m and n are relatively prime positive integers. Let ζ_m be a primitive m^{th} root of unity and let ζ_n be a primitive n^{th} root of unity. Prove that $\zeta_m\zeta_n$ is a primitive mn^{th} root of unity.

5. Prove there are only a finite number of roots of unity in any finite extension K of $\mathbb{Q}$.

6. For any prime p and any nonzero $a \in \mathbb{F}_p$ prove that $x^p - x + a$ is irreducible and separable over $\mathbb{F}_p$. [Hint: One approach is to prove first that if α is a root then $\alpha+1$ is also a root. Another approach is to suppose it's reducible and compute derivatives.]

7. Let k be a field and let $k(x)$ be the field of rational functions in the variable (or transcendental element) x with coefficients from k. Let $y \in k(x)$ be a fixed, nonconstant rational function, i.e., $y = \dfrac{P(x)}{Q(x)}$ for some relatively prime polynomials $P(x), Q(x) \in k[x]$, with $Q(x) \neq 0$. Note that
$$k(y) \subseteq k(x) \quad \text{and moreover} \quad k(x) = k(y)(x).$$
The purpose of this exercise is to compute the (finite) degree of the extension $k(x)/k(y)$; we do so by finding an irreducible polynomial in $k(y)[X]$ for which x is a root (here X is another variable, independent from x).

   (a) Explain why the polynomial $f(X) = P(X) - yQ(X)$ is a polynomial in the variable X with coefficients in $k(y)$ that has x as a root.

   (b) Show that the degree of $f(X)$, as a polynomial in X with coefficients in $k(y)$, is the maximum of the degrees of $P(x)$ and $Q(x)$ as polynomials in x (in particular, $f(X) \neq 0$). (Note: This shows x is algebraic over $k(y)$; and since $\infty = [k(x) : k] = [k(x) : k(y)][k(y) : k]$, we must have $[k(y) : k] = \infty$, i.e., y is transcendental over k too.)

   (c) Show that $f(X)$ is irreducible over $k(y)$. [Hint: Use (b) and Gauss' Lemma to show this polynomial is irreducible in $k(y)[X]$ if and only if it is irreducible in $k[y][X]$. Then note that we may view $k[y][X] = k[X][y]$ — why does this perspective help?]

   (d) Deduce from (a) to (c) that
$$\left[k(x) : k\left(\frac{P(x)}{Q(x)}\right)\right] = \max\left(\deg P(x), \deg Q(x)\right).$$
   An important special consequence is that $k(x) = k(y)$ if and only if $y \notin k$ and y is a quotient of polynomials of degree ≤ 1. Finite extensions of $k(x)$ are called *function fields* over k — they play a fundamental role in the study of curves over k, such as Riemann surfaces (when $k = \mathbb{C}$).