Math 252 Homework # 7
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1. Use Galois theory to prove that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are not isomorphic.

2. Determine the Galois group of the splitting field of $(x^2-2)(x^2-3)(x^2-5)$ over $\mathbb{Q}$. Determine *all* the subfields of the splitting field of this polynomial.

3. Determine the Galois group of the splitting field over $\mathbb{Q}$ of $x^4 - 14x^2 + 9$.

4. Prove that if the degree of the splitting field of $p(x)$ over $\mathbb{Q}$ is odd, then all the roots of $p(x)$ are real. (Do not assume $p(x)$ is irreducible.)

5. Show that $\mathbb{Q}(\sqrt{2+\sqrt{2}})$ is a cyclic quartic field, i.e., is a Galois extension of $\mathbb{Q}$ of degree 4 with cyclic Galois group.

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6. This exercise determines $\mathrm{Aut}(\mathbb{R}/\mathbb{Q})$.
   (a) Prove that any $\sigma \in \mathrm{Aut}(\mathbb{R}/\mathbb{Q})$ takes squares to squares and takes positive reals to positive reals. Conclude that $a < b$ implies $\sigma a < \sigma b$ for every $a, b \in \mathbb{R}$.
   (b) Prove that $-\frac{1}{m} < a - b < \frac{1}{m}$ implies $-\frac{1}{m} < \sigma a - \sigma b < \frac{1}{m}$ for every positive integer m. Conclude that σ is a continuous map on $\mathbb{R}$.
   (c) Prove that any continuous map on $\mathbb{R}$ which is the identity on $\mathbb{Q}$ is the identity map, hence $\mathrm{Aut}(\mathbb{R}/\mathbb{Q}) = 1$.

7. (a) Prove that $x^4 - 2x^2 - 2$ is irreducible over $\mathbb{Q}$.
   (b) Show the roots of this quartic are

$$\alpha_1 = \sqrt{1+\sqrt{3}} \qquad \alpha_3 = -\sqrt{1+\sqrt{3}}$$
$$\alpha_2 = \sqrt{1-\sqrt{3}} \qquad \alpha_4 = -\sqrt{1-\sqrt{3}}.$$

   (c) Let $K_1 = \mathbb{Q}(\alpha_1)$ and $K_2 = \mathbb{Q}(\alpha_2)$. Show that $K_1 \neq K_2$, and $K_1 \cap K_2 = \mathbb{Q}(\sqrt{3}) = F$.
   (d) Prove that K_1, K_2 and K_1K_2 are Galois over F with $\mathrm{Gal}(K_1K_2/F)$ the Klein 4-group. Write out the elements of $\mathrm{Gal}(K_1K_2/F)$ explicitly. Determine all the subgroups of the Galois group and give their corresponding fixed subfields of K_1K_2 containing F.
   (e) Prove that the splitting field of $x^4 - 2x^2 - 2$ over $\mathbb{Q}$ is of degree 8 with dihedral Galois group.