Math 252
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Homework # 8

1. This exercise determines the splitting field K for the polynomial $f(x) = x^6 - 2x^3 - 2$ over $\mathbb{Q}$.

   (a) Prove that $f(x)$ is irreducible over $\mathbb{Q}$ with roots the three cube roots of $1 \pm \sqrt{3}$.

   (b) Prove that K contains the field $\mathbb{Q}(\sqrt{-3})$ of 3^{rd} roots of unity and contains $\mathbb{Q}(\sqrt{3})$, hence contains the biquadratic field $F = \mathbb{Q}(i, \sqrt{3})$. Take the product of two of the roots in (a) to prove that K contains $\sqrt[3]{2}$ and conclude that K is an extension of the field $L = \mathbb{Q}(\sqrt[3]{2}, i, \sqrt{3})$.

   (c) Prove that $[L : \mathbb{Q}] = 12$ and that K is obtained from L by adjoining the cube root of an element in L, so that $[K : \mathbb{Q}] = 12$ or 36.

   (d) Prove that if $[K : \mathbb{Q}] = 12$ then $K = \mathbb{Q}(\sqrt[3]{2}, i, \sqrt{3})$ and that $\mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to the direct product of the cyclic group of order 2 and S_3. Prove that if $[K : \mathbb{Q}] = 12$ then there is a unique real cubic subfield in K, namely $\mathbb{Q}(\sqrt[3]{2})$.

   (e) Take the quotient of the two real roots in (a) to show that $\sqrt[3]{2+\sqrt{3}}$ and $\sqrt[3]{2-\sqrt{3}}$ (real roots) are both elements of K. Show that $\alpha = \sqrt[3]{2+\sqrt{3}} + \sqrt[3]{2-\sqrt{3}}$ is a real root of the irreducible cubic equation $x^3 - 3x - 4$ whose discriminant is $-2^2 3^4$. Conclude that the Galois closure of $\mathbb{Q}(\alpha)$ contains $\mathbb{Q}(i)$ so in particular $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\sqrt[3]{2})$.

   (f) Conclude from (e) that $G = \mathrm{Gal}(K/\mathbb{Q})$ is of order 36. Determine all the elements of G explicitly and in particular show that G is isomorphic to $S_3 \times S_3$.

   (g) Let $F = \mathbb{Q}(i, \sqrt{3})$, so K is Galois over F. Draw the lattice of all fields L with $F \subseteq L \subseteq K$, and draw the corresponding lattice of subgroups in $\mathrm{Gal}(K/F)$ (which is a *subgroup* of $\mathrm{Gal}(K/\mathbb{Q})$—what is its isomorphism type?)

2. Prove that the Galois group over $\mathbb{Q}$ of $x^6 - 4x^3 + 1$ is isomorphic to the dihedral group of order 12. [Observe that the two real roots are inverses of each other.]

3. Let k be the field with 4 elements, t a transcendental over k, $F = k(t^4 + t)$ and $K = k(t)$.

   (a) Show that $[K : F] = 4$. [You may quote results from previous homeworks.]

   (b) Show that K is separable over F.

   (c) Show that K is Galois over F.

   (d) Describe the lattice of subgroups of the Galois group and the corresponding lattice of subfields of K, giving each subfield in the form $k(r)$, for some rational function r.

4. Let K be a subfield of $\mathbb{C}$ maximal with respect to the property "$\sqrt{2} \notin K$." You may assume such a field K exists (it is easy to prove by Zorn's Lemma).

   (a) Show that $\mathbb{C}$ is algebraic over K.

   (b) Prove that every finite extension of K in $\mathbb{C}$ is Galois with Galois group a cyclic 2-group.

   (c) Deduce that $[\mathbb{C} : K]$ is countable (and not finite).