Math 252 Homework # 9
R. Foote

1. Find the rational canonical forms of

$$\begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} c & 0 & -1 \\ 0 & c & 1 \\ -1 & 1 & c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 422 & 465 & 15 & -30 \\ -420 & -463 & -15 & 30 \\ 840 & 930 & 32 & -60 \\ -140 & -155 & -5 & 12 \end{pmatrix}.$$

2. Find all similarity classes of 6×6 matrices over $\mathbb{Q}$ with minimal polynomial $(x+2)^2(x-1)$ (it suffices to give all lists of invariant factors and write out just a couple of their corresponding matrices).

3. Find all similarity classes of 6×6 matrices over $\mathbb{C}$ with characteristic polynomial $(x^4 - 1)(x^2 - 1)$.

4. Determine all possible rational canonical forms for a linear transformation with characteristic polynomial $x^2(x^2 + 1)^2$.

5. Determine which of the following matrices are similar:

$$\begin{pmatrix} -1 & 4 & -4 \\ 2 & -1 & 3 \\ 0 & -4 & 3 \end{pmatrix} \quad \begin{pmatrix} -3 & -4 & 0 \\ 2 & 3 & 0 \\ 8 & 8 & 1 \end{pmatrix} \quad \begin{pmatrix} -3 & 2 & -4 \\ 2 & 1 & 0 \\ 3 & -1 & 3 \end{pmatrix} \quad \begin{pmatrix} -1 & 4 & -4 \\ 0 & -3 & 2 \\ 0 & -4 & 3 \end{pmatrix}.$$

6. Determine the Jordan canonical forms for the following matrices:

$$\begin{pmatrix} 5 & 4 & 1 \\ -1 & 0 & 0 \\ -3 & -4 & 1 \end{pmatrix} \quad \begin{pmatrix} 3 & 4 & 2 \\ -2 & -3 & -1 \\ -4 & -4 & -3 \end{pmatrix}.$$

7. Verify for yourself that the matrices

$$A = \begin{pmatrix} -8 & -10 & -1 \\ 7 & 9 & 1 \\ 3 & 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 2 & -4 \\ 4 & -1 & 4 \\ 4 & -2 & 5 \end{pmatrix}$$

both have $(x-1)^2(x+1)$ as characteristic polynomial. Determine the Jordan canonical form for both matrices. Explain why one can be diagonalized and the other cannot.

8. Show that the characteristic polynomial of

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{pmatrix}$$

is a product of linear factors over $\mathbb{Q}$. Determine the rational and Jordan canonical forms for A over $\mathbb{Q}$.

9. Determine the Jordan canonical form for the matrix

$$\begin{pmatrix} 3 & 0 & -2 & -3 \\ 4 & -8 & 14 & -15 \\ 2 & -4 & 7 & -7 \\ 0 & 2 & -4 & 3 \end{pmatrix}.$$

10. Verify for yourself that the matrices

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 2 & -8 & -8 \\ -6 & -3 & 8 & 8 \\ -3 & -1 & 3 & 4 \\ 3 & 1 & -4 & -5 \end{pmatrix}$$

both have characteristic polynomial $(x-3)(x+1)^3$. Determine the Jordan canonical form for each matrix and explain whether or not they are similar.

11. (a) Find all similarity classes of 3×3 matrices A over $\mathbb{F}_2$ satisfying $A^6 = I$ (compare with the answer you computed over $\mathbb{Q}$ above).

(b) Find all similarity classes of 4×4 matrices B over $\mathbb{F}_2$ satisfying $B^{20} = I$.

12. Show that if $A^2 = A$ then A is similar to a diagonal matrix that has only 0's and 1's along the diagonal. (The entries of A may be from any field.)

13. Prove there are no 3×3 matrices A with entries from $\mathbb{Q}$ with $A^8 = I$ but $A^4 \neq I$.