MATH 3551 A — Abstract Algebra I
Practice Final Examination — SOLUTIONS

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Fall 2026 • Total: 200 points

About this practice exam. It was drafted with the help of Claude Opus 5, running in a Cowork harness, working from previous exams in this course. **It has not yet been timed or proofread.** It is meant to run well inside a single class period, with time to spare. *Please hunt for mistakes.* If you find a wrong answer, an ambiguous or unanswerable problem, a problem that leans on something we have not covered, or one that takes far longer than its point value suggests, tell me — in class, in office hours, or by email. I would much rather hear it from you now than print it on the real exam.

Problem 1. (60 points) **True or false; prove or disprove.**

(a) **TRUE.** $(1\,2\,3)(4\,5\,6\,7)$ has order $\operatorname{lcm}(3,4) = 12$ and uses $3 + 4 = 7$ points.

(b) **FALSE.** A homomorphism $\mathbb{Z}/12\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$ is determined by the image x of $\overline{1}$, subject to $12x \equiv 0 \pmod 8$, i.e. $4x \equiv 0 \pmod 8$, i.e. x even. Taking $x = \overline{2}$ gives a nontrivial homomorphism (with image of order 4).

(c) **TRUE.** $|(\mathbb{Z}/18\mathbb{Z})^\times| = \varphi(18) = 6$, and $\overline{5}$ has order 6: $5, 25 \equiv 7, 35 \equiv 17, 85 \equiv 13, 65 \equiv 11, 55 \equiv 1$.

(d) **TRUE.** This is D&F §4.2 Corollary 5: act on the p cosets to get $\pi\colon G \to S_p$ with $K = \ker \pi \le H$; then $|G : K|$ divides both $p!$ and $|G|$, and since p is the smallest prime divisor of $|G|$ this forces $|G : K| = p$, i.e. $K = H$.

(e) **FALSE.** $105 = 3 \cdot 5 \cdot 7$, and there is a nonabelian group $\mathbb{Z}/5\mathbb{Z} \times (\mathbb{Z}/7\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z})$ of that order (the semidirect product exists because $3 \mid |\mathrm{Aut}(\mathbb{Z}/7\mathbb{Z})| = 6$).

(f) **TRUE.** Choose the six moved points in $\binom{8}{6} = 28$ ways and arrange them in a cycle in $5! = 120$ ways: $28 \cdot 120 = 3360$.

(g) **TRUE.** An automorphism of V_4 permutes its three elements of order 2, and every permutation of them extends to an automorphism (any bijection between two-element generating sets of V_4 is an automorphism); so $\mathrm{Aut}(V_4) \cong S_3$.

(h) **TRUE.** $|A_5| = 60 = 2^2 \cdot 3 \cdot 5$ and $n_5 = 6$, so the normalizer of a Sylow 5-subgroup has order $60/6 = 10$. (It is the dihedral group D_{10} generated by a 5-cycle and a double transposition inverting it.)

(i) FALSE. A_4 has order 12 and $Z(A_4) = 1$, so its upper central series never leaves 1: A_4 is not nilpotent.

(j) FALSE. $36 = 2^2 \cdot 3^2$ gives $n_3 \in \{1, 4\}$. If the group were simple then $n_3 = 4$, and the conjugation action on the four Sylow 3-subgroups gives $G \to S_4$ with kernel a proper normal subgroup, hence trivial by simplicity; but then $36 \mid 24$, which is false.

(k) TRUE. $|Q_8| = 8 = 2^3$, and every finite p-group is nilpotent.

(l) FALSE. $Z(A_4) = 1$ (see item (i)).

(m) TRUE. They are $\mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, D_8 and Q_8.

(n) FALSE. D_8 is a nonabelian group of order 2^3; for odd p the Heisenberg group of upper unitriangular 3×3 matrices over $\mathbb{F}_p$ is nonabelian of order p^3.

(o) TRUE. $70 = 2 \cdot 5 \cdot 7$; the four groups are $\mathbb{Z}/70\mathbb{Z}$, $\mathbb{Z}/7\mathbb{Z} \times D_{10}$, $\mathbb{Z}/5\mathbb{Z} \times D_{14}$ and D_{70}.

(p) FALSE. D_8 is nonabelian, but $Z(D_8) = \langle r^2 \rangle$ and $D_8/\langle r^2 \rangle$ has order 4, hence is abelian. (What is true is that $G/Z(G)$ is never nontrivial *cyclic*.)

(q) TRUE. With D&F's convention $[a, b] = a^{-1}b^{-1}ab$ we get $[s, r] = s^{-1}r^{-1}sr = (sr^{-1}s)r = r \cdot r = r^2$, so $\langle r^2 \rangle \leq D_8'$; and $D_8/\langle r^2 \rangle$ has order 4 and is abelian, so $D_8' \leq \langle r^2 \rangle$.

(r) TRUE. $A_n' \trianglelefteq A_n$ and A_n is simple for $n \geq 5$, so A_n' is 1 or A_n; it is not 1, since A_n is nonabelian.

(s) TRUE. If $\overline{G} = G/N$ then $\gamma^i(\overline{G})$ is the image of $\gamma^i(G)$, so the lower central series of $\overline{G}$ reaches 1 at least as fast as that of G.

(t) FALSE. The number of abelian groups of order 2^5 is the number of partitions of 5, which is 7, not 6.

Grading note. Three points each: one for the circle, two for the reason. The discriminating items are (b), (d), (h), (p) and (t); (i) and (l) test the same fact twice on purpose, so a student who answers them inconsistently should be looked at. Accept any correct counterexample.

Problem 2. (20 points) **A permutation.**

(a) $|\tau| = \operatorname{lcm}(4, 2, 3) = 12$.

(b) $100 \equiv 0 \pmod 4$, $100 \equiv 0 \pmod 2$, $100 \equiv 1 \pmod 3$, so only the 3-cycle survives: $\tau^{100} = (6\,8\,9)$.

(c) A k-cycle is a product of $k-1$ transpositions, so τ is a product of $3+1+2 = 6$ transpositions: τ is **even**.

(d) Yes: both have cycle type $(4, 3, 2)$, and permutations of S_9 are conjugate exactly when they have the same cycle type. To build π, write the two permutations with their cycles in matching order and read off the correspondence:

$$\tau = (1\,2\,4\,7)(6\,8\,9)(3\,5), \qquad \rho = (1\,2\,3\,4)(5\,6\,7)(8\,9),$$

so take $\pi(1) = 1$, $\pi(2) = 2$, $\pi(4) = 3$, $\pi(7) = 4$, $\pi(6) = 5$, $\pi(8) = 6$, $\pi(9) = 7$, $\pi(3) = 8$, $\pi(5) = 9$; in cycle notation

$$\pi = (3\,8\,6\,5\,9\,7\,4).$$

Then $\pi\tau\pi^{-1} = (\pi1\,\pi2\,\pi4\,\pi7)(\pi6\,\pi8\,\pi9)(\pi3\,\pi5) = (1\,2\,3\,4)(5\,6\,7)(8\,9) = \rho$. ✓

(e) The class of τ has $\dfrac{9!}{4 \cdot 3 \cdot 2} = \dfrac{362880}{24} = 15120$ elements, so $|C_{S_9}(\tau)| = 362880/15120 = 24$.

Grading note. Part (d) is where the credit is: the conjugating permutation must be produced, and it must be checked or at least visibly read off from aligned cycle decompositions. Award 2 points for "same cycle type, hence conjugate" and 3 for a correct π. Any correct π is fine — there are 24 of them, one for each element of the centralizer times this one.

Problem 3. (20 points) **A group of order** 204**.**

(a) $n_{17} \mid 12$ and $n_{17} \equiv 1 \pmod{17}$. The divisors of 12 are $1, 2, 3, 4, 6, 12$, all less than 17, so only $n_{17} = 1$ is congruent to 1. Hence the Sylow 17-subgroup P is unique, therefore normal.

(b) Let $Q \in \operatorname{Syl}_3(G)$, of order 3. Since $P \trianglelefteq G$, PQ is a subgroup; $|P \cap Q|$ divides $\gcd(17, 3) = 1$, so $|PQ| = 17 \cdot 3 = 51$.

(c) First, every group K of order $12 = 2^2 \cdot 3$ has a normal Sylow subgroup: $n_3 \in \{1, 4\}$, and if $n_3 = 4$ then the four Sylow 3-subgroups contribute $4 \cdot 2 = 8$ elements of order 3, leaving $12 - 8 = 4$ elements, which must therefore be the unique Sylow 2-subgroup, so $n_2 = 1$.

Now $P \trianglelefteq G$ with $P \cong \mathbb{Z}/17\mathbb{Z}$ abelian, and G/P has order $204/17 = 12$. By the previous paragraph G/P has a normal subgroup M/P with M/P of order 3 or 4, and the corresponding quotients are of order 4 or 3; in either case we obtain a chain

$$1 \trianglelefteq P \trianglelefteq M \trianglelefteq G$$

with quotients $P \cong \mathbb{Z}/17\mathbb{Z}$, M/P of order 3 or 4, and G/M of order 4 or 3. Groups of order 3 and of order 4 are abelian, so all the factors are abelian and G is solvable. (Equivalently: P is solvable and G/P of order 12 is solvable, and an extension of a solvable group by a solvable group is solvable.)

(d) P is the unique subgroup of order 17, and every element of order 17 generates a subgroup of order 17, hence lies in P. So the elements of order 17 are the $17 - 1 = \boxed{16}$ non-identity elements of P.

Grading note. Part (c) is worth 7 and can be done either by exhibiting the chain or by citing the extension property of solvability (D&F §3.4 #8) — but in either case the order-12 lemma has to be established, and that is 4 of the 7 points. Expected error in (a): forgetting to check that all divisors of 12 are below 17, and instead writing "$n_{17} = 1$ or 18"; 18 is not a divisor of 12.

Problem 4. (20 points) **The groups of order 55.**

(a) $55 = 5 \cdot 11$, and $n_{11} \mid 5$ with $n_{11} \equiv 1 \pmod{11}$ forces $n_{11} = 1$. So the Sylow 11-subgroup $P \cong \mathbb{Z}/11\mathbb{Z}$ is normal.

(b) Let $Q \in \mathrm{Syl}_5(G)$, so $Q \cong \mathbb{Z}/5\mathbb{Z}$. Then $P \cap Q = 1$ (its order divides $\gcd(11,5) = 1$) and $|PQ| = 55 = |G|$, so $PQ = G$. Together with $P \trianglelefteq G$ this says G is the internal semidirect product $P \rtimes_\varphi Q$, where $\varphi(q)$ is conjugation by q.

(c) $\mathrm{Aut}(P) \cong (\mathbb{Z}/11\mathbb{Z})^\times$ is cyclic of order 10. A homomorphism $\varphi\colon \mathbb{Z}/5\mathbb{Z} \to \mathbb{Z}/10\mathbb{Z}$ sends a generator to an element of order dividing 5, and a cyclic group of order 10 has exactly one subgroup of order 5, hence exactly five such elements. So there are five homomorphisms φ.

- φ trivial gives $G \cong \mathbb{Z}/11\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \cong \mathbb{Z}/55\mathbb{Z}$.
- The four nontrivial φ all give isomorphic groups: if φ_1 and φ_2 are nontrivial then they have the same image (the unique subgroup of order 5 of $\mathrm{Aut}(P)$), so $\varphi_2 = \varphi_1 \circ \alpha$ for some $\alpha \in \mathrm{Aut}(Q)$, and precomposing with an automorphism of Q yields isomorphic semidirect products.

So up to isomorphism there are exactly **two** groups of order 55: the cyclic one and one nonabelian one.

(d) Let G be the nonabelian one. $Z(G) \trianglelefteq G$, so $|Z(G)| \in \{1, 5, 11, 55\}$. If $|Z(G)| = 11$ then $P \le Z(G)$ and the conjugation action of Q on P is trivial, making G abelian; if $|Z(G)| = 5$ then $G/Z(G)$ has order 11, hence is cyclic, hence G is abelian; and $|Z(G)| = 55$ says G is abelian outright. So $Z(G) = 1$. For n_5: $n_5 \mid 11$ and $n_5 \equiv 1 \pmod 5$, so $n_5 \in \{1, 11\}$; if $n_5 = 1$ then $Q \trianglelefteq G$ too and $G \cong P \times Q$ would be abelian. Hence $n_5 = 11$.

Grading note. As on the real exam, the trap in (c) is "five homomorphisms, so five groups". Award 2 points for $\mathrm{Aut}(\mathbb{Z}/11\mathbb{Z}) \cong \mathbb{Z}/10\mathbb{Z}$, 3 for the count of five homomorphisms, 3 for the argument that the four nontrivial ones give one isomorphism class. In (d), the elimination of the three nontrivial possibilities for $|Z(G)|$ is worth 2 and $n_5 = 11$ worth 2.

Problem 5. (20 points) **Abelian groups.**

(a) $36 = 2^2 \cdot 3^2$, $50 = 2 \cdot 5^2$, $8 = 2^3$, so $|A| = 36 \cdot 50 \cdot 8 = 14400 = 2^6 \cdot 3^2 \cdot 5^2$. The elementary divisors are

$$\underbrace{8,\ 4,\ 2}_{p=2}, \qquad \underbrace{9}_{p=3}, \qquad \underbrace{25}_{p=5}.$$

Three invariant factors; aligning from the largest and multiplying down the columns gives $8 \cdot 9 \cdot 25 = 1800$, then 4, then 2:

$$2 \mid 4 \mid 1800, \qquad A \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{1800}$$

(check: $2 \cdot 4 \cdot 1800 = 14400$).

(b) $p(6)\,p(2)\,p(2) = 11 \cdot 2 \cdot 2 = \boxed{44}$.

(c) The largest order of an element is the exponent, i.e. the largest invariant factor, 1800. Since $1800 = 2^3 \cdot 3^2 \cdot 5^2$: $200 = 2^3 \cdot 5^2$ divides 1800, so **yes**, there is an element of order 200; $16 = 2^4$ does not divide 1800, so **no** element of order 16.

(d) ($\Rightarrow$) If A is cyclic of order $n = \prod p_i^{e_i}$, then its Sylow p_i-subgroup is the unique subgroup of order $p_i^{e_i}$ of a cyclic group, hence cyclic.

($\Leftarrow$) Suppose each Sylow subgroup A_i is cyclic, say of order $p_i^{e_i}$, and pick a generator a_i. By the decomposition of a finite abelian group into its Sylow subgroups, $A \cong A_1 \times \cdots \times A_k$. The element $a = a_1 a_2 \cdots a_k$ has order $\operatorname{lcm}(p_1^{e_1}, \ldots, p_k^{e_k}) = \prod p_i^{e_i} = |A|$, since the $p_i^{e_i}$ are pairwise coprime. Hence $A = \langle a \rangle$ is cyclic.

Grading note. The alignment rule in (a) is the whole skill: $8, 4, 2$ against 9 and 25, padded with 1s. A common error is to produce two invariant factors rather than three. In (c) the phrase “the exponent is the largest invariant factor” should appear; a student who answers 14400 has confused order with exponent. Part (d) may quote the Sylow decomposition of a finite abelian group, but the coprime-lcm step must be there.

Problem 6. (20 points) **Solvable and nilpotent.**

(a) $\gamma^1(D_8) = D_8$. Next $\gamma^2(D_8) = [D_8, D_8] = \langle r^2 \rangle$ (because $[s, r] = s^{-1}r^{-1}sr = r^2$, and $D_8/\langle r^2\rangle$ has order 4, hence is abelian). Finally $\gamma^3(D_8) = [D_8, \langle r^2\rangle] = 1$, since $r^2 \in Z(D_8)$. So the lower central series is

$$D_8 \;\geq\; \langle r^2 \rangle \;\geq\; 1,$$

and D_8 is nilpotent of class 2.

(b) $A_4^{(1)} = A_4' = V_4$: indeed A_4/V_4 has order 3, hence is abelian, so $A_4' \leq V_4$; and $[(1\,2\,3), (1\,2)(3\,4)] = (1\,4)(2\,3) \neq 1$ lies in A_4', so $A_4' \neq 1$, and $A_4' \trianglelefteq A_4$ with V_4 the only normal subgroup of order 4 forces $A_4' = V_4$ (the only proper nontrivial normal subgroup of A_4 is V_4). Then $A_4^{(2)} = V_4' = 1$ since V_4 is abelian. The derived series is $A_4 \rhd V_4 \rhd 1$, so A_4 is solvable of derived length 2.

(c) In $D_{2n} = \langle r, s\rangle$ the subgroup $\langle r\rangle$ is cyclic of order n and has index 2, hence is normal, and $D_{2n}/\langle r\rangle \cong \mathbb{Z}/2\mathbb{Z}$. So $1 \trianglelefteq \langle r\rangle \trianglelefteq D_{2n}$ is a chain with abelian quotients ($\langle r\rangle \cong \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$), and D_{2n} is solvable.

(d) $Z(A_4) = 1$: a nonidentity element of A_4 is either a 3-cycle or a double transposition, and neither is central — e.g. $(1\,2\,3)$ does not commute with $(1\,2)(3\,4)$, and $(1\,2)(3\,4)$ does not commute with $(1\,2\,3)$. So the upper central series of A_4 is $1 = Z_0 = Z_1 = \cdots$ and never reaches A_4: A_4 is not nilpotent. (Equivalently: a finite nilpotent group has all Sylow subgroups normal, but $n_3(A_4) = 4$.)

Grading note. Part (b): 3 points for $A_4' \leq V_4$ (via the abelian quotient), 2 for the reverse. Part (d): either the centre argument or the Sylow argument is worth full credit, but "A_4 is not abelian so it is not nilpotent" is false reasoning and earns 0 — D_8 is nonabelian and nilpotent. In (c) watch for students who try to prove $\langle r\rangle$ normal by computing srs^{-1} only; that is fine, but the index-2 argument is faster.

Problem 7. (20 points) **The class equation.**

(a) For a finite group G, choosing representatives $g_1, \ldots, g_m$ of the conjugacy classes of size greater than 1,

$$|G| \;=\; |Z(G)| \;+\; \sum_{i=1}^{m} |G : C_G(g_i)|,$$

where $Z(G)$ collects exactly the elements whose class has size 1, and $|G : C_G(g_i)|$ is the size of the class of g_i (orbit–stabilizer for the conjugation action).

(b) $Z(D_8) = \{1, r^2\}$, and the remaining classes are $\{r, r^3\}$, $\{s, sr^2\}$ and $\{sr, sr^3\}$. So

$$8 = 1 + 1 + 2 + 2 + 2.$$

(c) $Z(A_4) = 1$. The three double transpositions form a single class of size 3, and the eight 3-cycles split into two classes of size 4 (the S_4-class of 3-cycles has 8 elements and $|C_{A_4}((1\,2\,3))| = 3$, so each A_4-class has $12/3 = 4$ elements). So

$$12 = 1 + 3 + 4 + 4.$$

(d) $Z(S_4) = 1$, and the classes are the cycle types: identity (1), transpositions (6), double transpositions (3), 3-cycles (8), 4-cycles (6). So

$$24 = 1 + 6 + 3 + 8 + 6.$$

Grading note. Part (c) is the discriminating one: the split of the 3-cycles into two A_4-classes is the point. A student who writes $12 = 1 + 3 + 8$ has given a correct partition of 12 into class-*type* sizes but the wrong class equation; award 2 of 5. In (b) and (d) the counts must be class sizes, each dividing $|G|$ — a quick sanity check students should perform themselves.

Problem 8. (20 points) **Isomorphic or not.**

(a) Both have order 60. Elementary divisors: $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$ gives $2, 3$ and $2, 5$, i.e. $\{2, 2\}$ at $p = 2$, $\{3\}$ at $p = 3$, $\{5\}$ at $p = 5$; $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/30\mathbb{Z}$ gives 2 and $2, 3, 5$, i.e. the same lists. Since the elementary divisors agree, the groups are isomorphic. (Both have invariant factors $2 \mid 30$.)

(b) D_{24} contains r of order 12. Every element of S_4 has cycle type 1^4, $(2, 1, 1)$, $(2, 2)$, $(3, 1)$ or (4), hence order at most 4. An isomorphism preserves orders of elements, so $D_{24} \not\cong S_4$.

(c) An internal semidirect product $G = H \rtimes K$ requires in particular $H \cap K = 1$. But every nontrivial subgroup of Q_8 contains -1: the subgroups of Q_8 are 1, $\{\pm 1\}$, $\langle i \rangle$, $\langle j \rangle$, $\langle k \rangle$ and Q_8, and every one of them except the trivial subgroup contains -1 (indeed -1 is the unique element of order 2, so any subgroup of even order contains it, and every nontrivial subgroup has even order by Lagrange). Hence for proper nontrivial $H, K \leq Q_8$ we always have $-1 \in H \cap K \neq 1$, and Q_8 is not such a semidirect product.

(d) Take $G = \mathbb{Z}/4\mathbb{Z}$ and $N = \langle \overline{2} \rangle \cong \mathbb{Z}/2\mathbb{Z}$. Then $N \trianglelefteq G$ (as G is abelian) and $G/N \cong \mathbb{Z}/2\mathbb{Z}$, but $N \times (G/N) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ has no element of order 4, whereas $G = \mathbb{Z}/4\mathbb{Z}$ does. So $G \not\cong N \times (G/N)$.

Grading note. Parts (b) and (d) are one-liners using the element-order invariant and should be quick. Part (c) is the interesting one: the key observation is that -1 lies in every nontrivial subgroup, which is a consequence of -1 being the unique element of order 2; award 4 of 6 for that observation alone. Students who instead try to enumerate all homomorphisms $K \to \mathrm{Aut}(H)$ will not finish and should be given credit only for correct partial work.