MATH 3551 A — Abstract Algebra I
Practice Final Examination

Taylor Dupuy, University of Vermont • Fall 2026

For practice — the final is Thursday, December 17, 2026 • Time limit: 2 hours 45 minutes • Total: 200 points

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NAME (please print): ____

Directions. This exam is **closed book**: no textbook, no notes, no calculators, no phones, no other devices, and no assistance of any kind. Write your answers on the exam itself; use the backs of pages if you need more room, and say clearly when you have done so. *Show your work and name the theorems you use* — "by Lagrange's Theorem, $|H|$ divides 24" earns nearly all of the credit, an unjustified correct number earns little. Point values are marked on each problem. Good luck.

About this practice exam. It was drafted with the help of Claude Opus 5, running in a Cowork harness, working from previous exams in this course. **It has not yet been timed or proofread.** It is meant to run well inside a single class period, with time to spare. *Please hunt for mistakes.* If you find a wrong answer, an ambiguous or unanswerable problem, a problem that leans on something we have not covered, or one that takes far longer than its point value suggests, tell me — in class, in office hours, or by email. I would much rather hear it from you now than print it on the real exam.

Problem 1. (60 points) **True or false; prove or disprove.**

Circle **T** or **F**, and then give a one-line reason. Three points each.

(a) **T** **F** There is an element of order 12 in the symmetric group S_7.

(b) **T** **F** Every group homomorphism $\mathbb{Z}/12\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$ is trivial.

(c) **T** **F** The group $(\mathbb{Z}/18\mathbb{Z})^\times$ is cyclic.

(d) **T** **F** If $H \leq G$ is a subgroup whose index is the smallest prime dividing $|G|$, then $H \trianglelefteq G$.

(e) **T** **F** Every group of order 105 is abelian.

(f) **T** **F** The number of 6-cycles in S_8 is 3360.

(g) **T** **F** $\mathrm{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \cong S_3$.

(h) **T** **F** A_5 contains a subgroup of order 10.

(i) **T** **F** Every group of order 12 is nilpotent.

(j) **T** **F** There is a simple group of order 36.

(k) **T** **F** The quaternion group Q_8 is nilpotent.

(l) **T** **F** The alternating group A_4 is nilpotent.

(m) **T** **F** There are exactly five groups of order 8 up to isomorphism.

(n) **T** **F** Every group of order p^3, p prime, is abelian.

(o) **T** **F** There are exactly four groups of order 70 up to isomorphism.

(p) **T** **F** If G is nonabelian then $G/Z(G)$ is nonabelian.

(q) **T** **F** The commutator subgroup of D_8 is $\langle r^2 \rangle$.

(r) **T** **F** For $n \geq 5$ the group A_n is perfect, i.e. $A_n' = A_n$.

(s) **T** **F** Every quotient group of a nilpotent group is nilpotent.

(t) **T** **F** There are exactly six abelian groups of order 2^5 up to isomorphism.

Problem 2. (20 points) **A permutation.**

Let $\tau = (1\,2\,4\,7)(3\,5)(6\,8\,9) \in S_9$.

(a) (3 points) Determine $|\tau|$.

(b) (4 points) Give the cycle decomposition of τ^{100}.

(c) (3 points) Is τ even or odd? Justify.

(d) (5 points) Is τ conjugate in S_9 to $\rho = (1\,2\,3\,4)(5\,6\,7)(8\,9)$? If so, exhibit an explicit $\pi \in S_9$ with $\pi\tau\pi^{-1} = \rho$.

(e) (5 points) Determine $|C_{S_9}(\tau)|$.

Problem 3. (20 points) **A group of order 204.**

Let G be a group of order $|G| = 204 = 2^2 \cdot 3 \cdot 17$.

(a) (5 points) Prove that G has a normal Sylow 17-subgroup P.

(b) (5 points) Prove that G has a subgroup of order 51.

(c) (7 points) Prove that G is solvable. *(You may want to show first that every group of order* 12 *has a normal Sylow subgroup.)*

(d) (3 points) How many elements of order 17 does G have?

Problem 4. (20 points) **The groups of order 55.**

(a) (4 points) Prove that a group G of order 55 has a normal Sylow 11-subgroup P.

(b) (4 points) Prove that $G \cong P \rtimes_\varphi Q$ for a Sylow 5-subgroup Q and a homomorphism $\varphi\colon Q \to \mathrm{Aut}(P)$.

(c) (8 points) Determine all such φ and conclude that there are exactly two groups of order 55 up to isomorphism.

(d) (4 points) For the nonabelian group of order 55, determine $Z(G)$ and n_5.

Problem 5. (20 points) **Abelian groups.**

Let $A = \mathbb{Z}_{36} \times \mathbb{Z}_{50} \times \mathbb{Z}_{8}$.

(a) (8 points) Determine $|A|$ in prime-power form, the elementary divisors of A, and the invariant factors of A.

(b) (4 points) How many abelian groups are there of order $|A|$? (Do not list them.)

(c) (4 points) What is the largest order of an element of A? Does A have an element of order 200? Of order 16?

(d) (4 points) Prove that a finite abelian group is cyclic if and only if all of its Sylow subgroups are cyclic.

Problem 6. (20 points) **Solvable and nilpotent.**

(a) (5 points) Compute the lower central series of D_8 and give the nilpotence class of D_8.

(b) (5 points) Compute the derived series of A_4 and conclude that A_4 is solvable.

(c) (5 points) Prove that D_{2n} is solvable for every $n \geq 3$.

(d) (5 points) Prove that A_4 is not nilpotent.

Problem 7. (20 points) **The class equation.**

(a) (4 points) State the class equation of a finite group G, explaining what each term means.

(b) (5 points) Determine the class equation of D_8.

(c) (5 points) Determine the class equation of A_4.

(d) (6 points) Determine the class equation of S_4.

Problem 8. (20 points) **Isomorphic or not.**

(a) (5 points) Prove that $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/30\mathbb{Z}$.

(b) (5 points) Prove that D_{24} (the dihedral group of order 24) is not isomorphic to S_4.

(c) (6 points) Prove that Q_8 is not the internal semidirect product $H \rtimes K$ of two proper nontrivial subgroups $H, K \leq Q_8$.

(d) (4 points) Give, with justification, an example of a group G and a normal subgroup $N \trianglelefteq G$ such that $G \ncong N \times (G/N)$.

End of practice exam. Total: $60 + 7 \times 20 = 200$ points.