MATH 3551 A — Abstract Algebra I
Practice Midterm 1 — SOLUTIONS

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About this practice exam. It was drafted with the help of Claude Opus 5, running in a Cowork harness, working from previous exams in this course. **It has not yet been timed or proofread.** It is meant to run well inside a single class period, with time to spare. *Please hunt for mistakes.* If you find a wrong answer, an ambiguous or unanswerable problem, a problem that leans on something we have not covered, or one that takes far longer than its point value suggests, tell me — in class, in office hours, or by email. I would much rather hear it from you now than print it on the real exam.

Problem 1. (30 points) **True or false; prove or disprove.**

(a) **FALSE.** The order of a permutation is the lcm of its cycle lengths, so we would need a partition of at most 9 whose parts have lcm 18. Since $18 = 2{\cdot}3^2$, such a partition must contain a part divisible by 9 and a part divisible by 2; the smallest possibility is $9 + 2 = 11 > 9$. No such element exists.

(b) **TRUE.** $|xy| = |x||y|$ for rational x, y, and the operation on $\mathbb{Q}^\times$ is multiplication, so $\varphi(xy) = \varphi(x)\varphi(y)$.

(c) **TRUE.** $D_{16} = D_{2n}$ with $n = 8$. No reflection is central (if sr^i were central then $r(sr^i)r^{-1} = sr^{i-2} = sr^i$ forces $r^2 = 1$, false), and r^m is central iff $r^m s = sr^m$, i.e. $r^m = r^{-m}$, i.e. $8 \mid 2m$, i.e. $m \in \{0, 4\}$. So $Z(D_{16}) = \{1, r^4\}$.

(d) **TRUE.** This is the multiplicativity of the index: choose coset representatives $g_1, \ldots, g_a$ for K in G and $k_1, \ldots, k_b$ for H in K; then the $g_i k_j$ are a complete, irredundant set of representatives for H in G.

(e) **FALSE.** The Klein four-group $V_4 = \{1, a, b, ab\}$ has three proper nontrivial subgroups, each of order 2 and hence cyclic (and the trivial subgroup is cyclic), yet V_4 is not cyclic — it has no element of order 4. (S_3 is another counterexample.)

(f) **TRUE.** 1 has finite order; if $|x| = m$ and $|y| = n$ then $(xy^{-1})^{mn} = x^{mn}y^{-mn} = 1$ using commutativity, so xy^{-1} has finite order. By the subgroup criterion this is a subgroup. (Commutativity is essential; in an infinite nonabelian group the set of torsion elements need not be a subgroup.)

(g) **FALSE.** $(\mathbb{Z}/15\mathbb{Z})^\times$ has order $\varphi(15) = 8$, but every element satisfies $x^4 = 1$: indeed $\overline{2}^4 = \overline{16} = \overline{1}$, $\overline{4}^2 = \overline{1}$, $\overline{7}^4 = \overline{1}$, $\overline{11}^2 = \overline{1}$, $\overline{13}^4 = \overline{1}$, $\overline{14}^2 = \overline{1}$, $\overline{8}^4 = \overline{1}$. So there is no element of order 8.

(h) **TRUE.** Choose the 3 moved points in $\binom{6}{3} = 20$ ways and arrange them in a cycle in $(3-1)! = 2$ ways: $20 \cdot 2 = 40$.

(i) **FALSE.** Take $y = x$: then $xy = x^2 = 1$, which has order 1, not 2. (The correct statement is $|xy| \mid \mathrm{lcm}(|x|, |y|) = 2$.)

(j) **FALSE.** $(\mathbb{Z}/8\mathbb{Z})^\times = \{\overline{1}, \overline{3}, \overline{5}, \overline{7}\}$ and each of $\overline{3}, \overline{5}, \overline{7}$ squares to $\overline{1}$, so every non-identity element has order 2; but $\mathbb{Z}/4\mathbb{Z}$ has an element of order 4. An isomorphism preserves orders of elements, so they are not isomorphic. (In fact $(\mathbb{Z}/8\mathbb{Z})^\times \cong V_4$.)

Grading note. The reason is worth two of the three points, and it has to be the right kind of reason: a counterexample for a false statement, a theorem or computation for a true one. Expected errors: (a) students who answer T because $18 \leq 9!$ or because "S_9 is big"; (e) students who cannot produce V_4 under time pressure; (i) students who miss the degenerate case $y = x$ — this is the item that separates people who read the hypotheses from people who pattern-match.

Problem 2. (20 points) **Inside $\mathbb{Z}/72\mathbb{Z}$.**

In $\mathbb{Z}/72\mathbb{Z}$ the order of $\overline{k}$ is $72/\gcd(k,72)$, and $\langle\overline{k}\rangle = \langle\overline{d}\rangle$ where $d = \gcd(k,72)$.

(a) $\gcd(30,72) = 6$, so $|\overline{30}| = 72/6 = 12$.

(b) $\langle\overline{30}\rangle = \langle\overline{6}\rangle$, the unique subgroup of order 12:

$$\{\overline{0},\overline{6},\overline{12},\overline{18},\overline{24},\overline{30},\overline{36},\overline{42},\overline{48},\overline{54},\overline{60},\overline{66}\}.$$

(c) $\varphi(72) = \varphi(2^3)\varphi(3^2) = 4\cdot 6 = 24$.

(d) $\gcd(18,72) = 18$, so $\langle\overline{18}\rangle$ is the subgroup of order $72/18 = 4$, namely $\{\overline{0},\overline{18},\overline{36},\overline{54}\}$. Now $\langle\overline{k}\rangle = \langle\overline{18}\rangle$ if and only if $\overline{k}$ is one of the $\varphi(4) = 2$ generators of that cyclic group of order 4, i.e. $k \in \{18, 54\}$. (Equivalently: $\gcd(k,72) = 18$.)

Grading note. Award the points for the *method*: $|\overline{k}| = n/\gcd(k,n)$ in (a), $\langle\overline{k}\rangle = \langle\overline{\gcd(k,n)}\rangle$ in (b) and (d), $\varphi(n)$ in (c). The expected error in (d) is to list all four elements of $\langle\overline{18}\rangle$ rather than only its two generators; that is worth 4 of the 7 points. A student who answers (d) by testing all 72 values gets full credit if the answer is right.

Problem 3. (15 points) **Two permutations.**

(a) $|\sigma| = \mathrm{lcm}(3,2,4) = 12$.

(b) Apply τ first, then σ: $1\mapsto 2\mapsto 5$, $5\mapsto 1\mapsto 4$, $4\mapsto 5\mapsto 2$, $2\mapsto 3\mapsto 6$, $6\mapsto 6\mapsto 8$, $8\mapsto 8\mapsto 9$, $9\mapsto 9\mapsto 3$, $3\mapsto 4\mapsto 7$, $7\mapsto 7\mapsto 1$. Hence

$$\sigma\tau = (1\,5\,4\,2\,6\,8\,9\,3\,7),$$

a 9-cycle, so $|\sigma\tau| = 9$.

(c) The point 3 lies on the 4-cycle $(3\,6\,8\,9)$, so σ^k fixes 3 if and only if $4 \mid k$. For $0 \le k < 12$ this gives $k \in \{0,4,8\}$.

(d) $\sigma^{-1} = (1\,7\,4)(2\,5)(3\,9\,8\,6)$.

Grading note. (b) is the part that takes time and it is where the composition order matters; a student who computes $\tau\sigma$ instead gets $(1\,5\,3\,8\,9\,6\,2\,4\,7)$ or similar — award 3 of the 6 points for a correct computation in the wrong order, provided the convention used is stated. In (c) the intended observation is that only the cycle containing 3 matters; students who compute all twelve powers of σ get full credit but will not have time for problem 5.

Problem 4. (15 points) **Centralizer and normalizer in the quaternion group.**

(a) $A = \langle i \rangle$, so $C_G(A) = C_G(i)$. Certainly ± 1 (central) and $\pm i$ (powers of i) commute with i. On the other hand $ji = -k$ while $ij = k$, so j does not commute with i; the same computation applies to $-j, k, -k$. Hence

$$C_G(A) = \{1, -1, i, -i\} = A.$$

(b) $jij^{-1} = -i \in A$, and $j(-1)j^{-1} = -1 \in A$, so $jAj^{-1} = A$ and $j \in N_G(A)$. Since $A \leq N_G(A)$ as well and $|N_G(A)|$ divides 8 and is more than 4, we get $N_G(A) = Q_8 = G$. (Alternatively: $|G : A| = 2$, and every subgroup of index 2 is normal.)

(c) $|G : A| = 8/4 = 2$, and the two left cosets are

$$A = \{1, -1, i, -i\}, \qquad jA = \{j, -j, ji, -ji\} = \{j, -j, -k, k\}.$$

They are disjoint, each of size 4, and their union is G: $2 \cdot 4 = 8 = |G|$, as Lagrange requires.

Grading note. As in the real exam, the point is that $C_G(A) \neq N_G(A)$: the element j inverts i rather than fixing it. A student who reports $C_G(A) = G$ "because Q_8 is almost abelian" has missed the entire problem. Give full credit in (b) for the index-2 shortcut. In (c) accept $\{j, -j, k, -k\}$ in any order; note $ji = -k$, so a student who writes $jA = \{j, -j, k, -k\}$ without computing ji has still got the right set.

Problem 5. (20 points) **The center.**

(a) $1 \in Z(G)$. If $z, w \in Z(G)$ and $g \in G$ then $(zw^{-1})g = zw^{-1}g = zgw^{-1} = gzw^{-1} = g(zw^{-1})$ (using $w^{-1}g = gw^{-1}$, which follows from $wg = gw$), so $zw^{-1} \in Z(G)$ and $Z(G) \leq G$. For normality, if $z \in Z(G)$ and $g \in G$ then $gzg^{-1} = zgg^{-1} = z \in Z(G)$. Hence $Z(G) \trianglelefteq G$.

(b) Write $Z = Z(G)$ and suppose $G/Z = \langle xZ \rangle$ for some $x \in G$. Let $g, h \in G$. Then $gZ = x^a Z$ and $hZ = x^b Z$ for some integers a, b, so we may write

$$g = x^a z_1, \qquad h = x^b z_2 \qquad \text{with } z_1, z_2 \in Z.$$

Since z_1, z_2 commute with everything,

$$gh = x^a z_1 x^b z_2 = x^{a+b} z_1 z_2 = x^{b+a} z_2 z_1 = x^b z_2 x^a z_1 = hg.$$

Thus G is abelian.

(c) Suppose $|G : Z(G)| = p$ is prime. Then $G/Z(G)$ has prime order, hence is cyclic (every group of prime order is cyclic, by Lagrange). By (b), G is abelian, so $Z(G) = G$ and $|G : Z(G)| = 1 \neq p$ — a contradiction. Therefore $|G : Z(G)|$ is never prime.

Grading note. Part (b) is the theorem (D&F §3.1 Exercise 36) and is worth ten points; the essential step is writing $g = x^a z_1$ with z_1 *central*, not merely $gZ = x^a Z$. A proof that stops at "gZ and hZ commute in G/Z" has proved only that G/Z is abelian, which was a hypothesis; award at most 3 points. Part (c) is four lines and should not be skipped: the contradiction is that G abelian forces $Z(G) = G$. Expected error in (a): proving $Z(G)$ is a subgroup and forgetting to prove normality, or vice versa (the two halves are worth 3 and 2 points respectively).