MATH 3551 A — Abstract Algebra I
Practice Midterm 1 — Version 2 — SOLUTIONS

Taylor Dupuy, University of Vermont • Fall 2026

Fall 2026 • Total: 100 points

Version 2 — September 28, 2026. Generated by GPT-5.6 Sol in the Codex desktop harness. Please notify Professor Dupuy if something looks off.

What changed from Version 1. The coverage now stops before §1.7. One true/false item and Problem 4 were replaced because §2.2 and Chapter 3 are not on Midterm 1. Selected material from §§0.1–0.3 has been added to make the foundations easier to practice, and that material is included on the exam. All other questions are unchanged. Version 1 remains available on the course webpage so that the changes can be compared directly.

This practice exam targets about thirty minutes of work in a fifty-minute period. Please report any error, ambiguity, out-of-coverage question, or item that takes much longer than its point value suggests.

Problem 1. (30 points) **True or false; prove or disprove.**

(a) **FALSE.** The order of a permutation is the lcm of its cycle lengths, so we would need a partition of at most 9 whose parts have lcm 18. Since $18 = 2\cdot 3^2$, such a partition must contain a part divisible by 9 and a part divisible by 2; the smallest possibility is $9 + 2 = 11 > 9$. No such element exists.

(b) **TRUE.** $|xy| = |x||y|$ for rational x, y, and the operation on $\mathbb{Q}^\times$ is multiplication, so $\varphi(xy) = \varphi(x)\varphi(y)$.

(c) **FALSE.** A class $\overline{a}$ is invertible in $\mathbb{Z}/n\mathbb{Z}$ exactly when $\gcd(a, n) = 1$. Here $\gcd(6, 15) = 3$, so $\overline{6}$ has no multiplicative inverse modulo 15.

(d) **FALSE.** $(\mathbb{Z}/15\mathbb{Z})^\times$ has order $\varphi(15) = 8$, but every element satisfies $x^4 = 1$: indeed $\overline{2}^4 = \overline{16} = \overline{1}$, $\overline{4}^2 = \overline{1}$, $\overline{7}^4 = \overline{1}$, $\overline{11}^2 = \overline{1}$, $\overline{13}^4 = \overline{1}$, $\overline{14}^2 = \overline{1}$, $\overline{8}^4 = \overline{1}$. So there is no element of order 8.

(e) **TRUE.** Choose the 3 moved points in $\binom{6}{3} = 20$ ways and arrange them in a cycle in $(3-1)! = 2$ ways: $20 \cdot 2 = 40$.

(f) **FALSE.** $(\mathbb{Z}/8\mathbb{Z})^\times = \{\overline{1}, \overline{3}, \overline{5}, \overline{7}\}$ and each of $\overline{3}, \overline{5}, \overline{7}$ squares to $\overline{1}$, so every non-identity element has order 2; but $\mathbb{Z}/4\mathbb{Z}$ has an element of order 4. An isomorphism preserves orders of elements, so they are not isomorphic. (In fact $(\mathbb{Z}/8\mathbb{Z})^\times \cong V_4$.)

Grading note. Most of the credit here is in the *reason*, and the reason must be specific: a named counterexample, the actual permutation, or the actual count. A circled answer with no reason earns two of the five points; a wrong circle with a reason that is right about the underlying fact earns the three reason-points. Give full credit for a correct counterexample even if it is not the one printed above. The parenthetical remarks in each answer flag the error to expect on that item.

Problem 2. (25 points) **Inside $\mathbb{Z}/72\mathbb{Z}$.**

In $\mathbb{Z}/72\mathbb{Z}$ the order of $\overline{k}$ is $72/\gcd(k,72)$, and $\langle\overline{k}\rangle = \langle\overline{d}\rangle$ where $d = \gcd(k,72)$.

(a) $\gcd(30,72) = 6$, so $|\overline{30}| = 72/6 = 12$.

(b) $\langle\overline{30}\rangle = \langle\overline{6}\rangle$, the unique subgroup of order 12:

$$\{\overline{0},\overline{6},\overline{12},\overline{18},\overline{24},\overline{30},\overline{36},\overline{42},\overline{48},\overline{54},\overline{60},\overline{66}\}.$$

(c) $\varphi(72) = \varphi(2^3)\varphi(3^2) = 4\cdot 6 = 24$.

(d) $\gcd(18,72) = 18$, so $\langle\overline{18}\rangle$ is the subgroup of order $72/18 = 4$, namely $\{\overline{0},\overline{18},\overline{36},\overline{54}\}$. Now $\langle\overline{k}\rangle = \langle\overline{18}\rangle$ if and only if $\overline{k}$ is one of the $\varphi(4) = 2$ generators of that cyclic group of order 4, i.e. $k \in \{18, 54\}$. (Equivalently: $\gcd(k,72) = 18$.)

> **Grading note.** Award the points for the *method*: $|\overline{k}| = n/\gcd(k,n)$ in (a), $\langle\overline{k}\rangle = \langle\overline{\gcd(k,n)}\rangle$ in (b) and (d), $\varphi(n)$ in (c). The expected error in (d) is to list all four elements of $\langle\overline{18}\rangle$ rather than only its two generators; that is worth 4 of the 7 points. A student who answers (d) by testing all 72 values gets full credit if the answer is right.

Problem 3. (25 points) **Two permutations.**

(a) $|\sigma| = \mathrm{lcm}(3,2,4) = 12$.

(b) Apply τ first, then σ: $1 \mapsto 2 \mapsto 5$, $5 \mapsto 1 \mapsto 4$, $4 \mapsto 5 \mapsto 2$, $2 \mapsto 3 \mapsto 6$, $6 \mapsto 6 \mapsto 8$, $8 \mapsto 8 \mapsto 9$, $9 \mapsto 9 \mapsto 3$, $3 \mapsto 4 \mapsto 7$, $7 \mapsto 7 \mapsto 1$. Hence

$$\sigma\tau = (1\,5\,4\,2\,6\,8\,9\,3\,7),$$

a 9-cycle, so $|\sigma\tau| = 9$.

(c) The point 3 lies on the 4-cycle $(3\,6\,8\,9)$, so σ^k fixes 3 if and only if $4 \mid k$. For $0 \le k < 12$ this gives $k \in \{0,4,8\}$.

(d) $\sigma^{-1} = (1\,7\,4)(2\,5)(3\,9\,8\,6)$.

> **Grading note.** (b) is the part that takes time and it is where the composition order matters; a student who computes $\tau\sigma$ instead gets $(1\,5\,3\,8\,9\,6\,2\,4\,7)$ or similar — award 3 of the 6 points for a correct computation in the wrong order, provided the convention used is stated. In (c) the intended observation is that only the cycle containing 3 matters; students who compute all twelve powers of σ get full credit but will not have time for problem 5.

Problem 4. (20 points) **Subgroups of the quaternion group.**

(a) The identity has order 1, the element -1 has order 2, and each of $\pm i, \pm j, \pm k$ has order 4 because its square is -1.

(b)

$$\langle i \rangle = \{1, -1, i, -i\}, \qquad \langle j \rangle = \{1, -1, j, -j\}, \qquad \langle i, j \rangle = Q_8,$$

since $ij = k$ and the subgroup generated by i and j therefore contains all eight displayed elements.

(c) The six subgroups are

$$\{1\}, \quad \{1, -1\}, \quad \langle i \rangle, \quad \langle j \rangle, \quad \langle k \rangle, \quad Q_8.$$

Every nonidentity subgroup contains either -1 alone or one of the six order-4 elements; the latter generate exactly the three order-4 subgroups listed above, so the list is complete.

(d) The group D_8 has five elements of order 2 (the four reflections and r^2), whereas Q_8 has exactly one, namely -1. An isomorphism preserves orders of elements, so $Q_8 \not\cong D_8$.

Grading note. In (b), the point is that two generators produce all of Q_8; award three of the five points for the three correct sets even without an explanation. In (c), the common omission is $\{1, -1\}$. In (d), any invariant that separates the groups earns full credit, but the count of elements of order 2 is the expected one-line argument.