MATH 3551 A — Abstract Algebra I
Practice Midterm 1

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For practice — Midterm 1 is Monday, October 12, 2026 • Time limit: 50 minutes • Total: 100 points

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NAME (please print): ______________________________________

Directions. This exam is **closed book**: no textbook, no notes, no calculators, no phones, no other devices, and no assistance of any kind. Write your answers on the exam itself; use the backs of pages if you need more room, and say clearly when you have done so. *Show your work and name the theorems you use* — "by Lagrange's Theorem, $|H|$ divides 24" earns nearly all of the credit, an unjustified correct number earns little. Point values are marked on each problem. Good luck.

About this practice exam. It was drafted with the help of Claude Opus 5, running in a Cowork harness, working from previous exams in this course. **It has not yet been timed or proofread.** It is meant to run well inside a single class period, with time to spare. *Please hunt for mistakes.* If you find a wrong answer, an ambiguous or unanswerable problem, a problem that leans on something we have not covered, or one that takes far longer than its point value suggests, tell me — in class, in office hours, or by email. I would much rather hear it from you now than print it on the real exam.

Problem 1. (30 points) **True or false; prove or disprove.**

Circle **T** or **F**, and then give a one-line reason: a theorem, a short computation, or an explicit counterexample. Three points each: one for the circle, two for the reason.

(a) T F There is an element of order 18 in the symmetric group S_9.

(b) T F The map $\varphi\colon \mathbb{Q}^\times \to \mathbb{Q}^\times$ defined by $\varphi(x) = |x|$ is a group homomorphism.

(c) T F The center of the dihedral group D_{16} of order 16 is $\{1, r^4\}$.

(d) T F If $H \leq K \leq G$ and $|G : H|$ is finite, then $|G : H| = |G : K| \cdot |K : H|$.

(e) T F If every proper subgroup of a group G is cyclic, then G is cyclic.

(f) T F The set of elements of finite order in an abelian group G is a subgroup of G.

(g) T F The group $(\mathbb{Z}/15\mathbb{Z})^{\times}$ is cyclic.

(h) T F The number of 3-cycles in S_6 is 40.

(i) T F If $x, y \in G$ satisfy $|x| = |y| = 2$ and $xy = yx$, then $|xy| = 2$.

(j) T F The groups $\mathbb{Z}/4\mathbb{Z}$ and $(\mathbb{Z}/8\mathbb{Z})^{\times}$ are isomorphic.

Problem 2. (20 points) **Inside $\mathbb{Z}/72\mathbb{Z}$.**

Work in the additive group $\mathbb{Z}/72\mathbb{Z}$.

(a) (4 points) Find the order of $\overline{30}$.

(b) (5 points) List the elements of the subgroup $\langle\overline{30}\rangle$.

(c) (4 points) How many generators does $\mathbb{Z}/72\mathbb{Z}$ have?

(d) (7 points) Determine every k with $0 \leq k < 72$ for which $\langle\overline{k}\rangle = \langle\overline{18}\rangle$.

Problem 3. (15 points) **Two permutations.**

In S_9 let

$$\sigma = (1\,4\,7)(2\,5)(3\,6\,8\,9), \qquad \tau = (1\,2\,3\,4\,5).$$

(Permutations act on the left, so $\sigma\tau$ means: first apply τ, then apply σ.)

(a) (4 points) What is $|\sigma|$?

(b) (6 points) Find the cycle decomposition of $\sigma\tau$, and its order.

(c) (3 points) For which k with $0 \le k < 12$ does σ^k fix the point 3?

(d) (2 points) Give the cycle decomposition of σ^{-1}.

Problem 4. (15 points) **Centralizer and normalizer in the quaternion group.**

Let $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, with the usual relations $i^2 = j^2 = k^2 = -1$, $ij = k$, $ji = -k$, and let $A = \langle i \rangle = \{1, -1, i, -i\}$.

(a) (6 points) Compute $C_G(A)$. Justify your answer.

(b) (6 points) Compute $N_G(A)$. Justify your answer.

(c) (3 points) List the left cosets of A in G and check Lagrange's Theorem for this pair.

Problem 5. (20 points) **The center.**

Let G be a group with center $Z(G) = \{z \in G : zg = gz \text{ for all } g \in G\}$.

(a) (5 points) Prove that $Z(G) \trianglelefteq G$.

(b) (10 points) Prove that if the quotient group $G/Z(G)$ is cyclic, then G is abelian.

(c) (5 points) Deduce that if G is a finite group, then $|G : Z(G)|$ is never a prime number.

End of practice exam. Total: $30 + 20 + 15 + 15 + 20 = 100$ points.