MATH 3551 A — Abstract Algebra I
Practice Midterm 2 — Version A — SOLUTIONS

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Problem 1. (20 points) **True or false; prove or disprove.**

(a) TRUE. A quotient of an abelian group is abelian because the quotient map is a homomorphism and images of commuting elements commute.

(b) FALSE. $N_G(H) = G$ holds exactly when H is normal. For example, a subgroup generated by a transposition in S_3 is not normal, so its normalizer is smaller than S_3.

(c) TRUE. If $k \in \ker \varphi$, then for every $g \in G$, $\varphi(gkg^{-1}) = \varphi(g)1\varphi(g)^{-1} = 1$, so $gkg^{-1} \in \ker \varphi$.

(d) FALSE. $\overline{4}$ has additive order $12/\gcd(12,4) = 3$, so its subgroup has order 3.

Grading note. Award 2 points for the correct circle and 3 for a valid reason. Accept any correct theorem or concise argument.

Problem 2. (30 points) **A quotient of Q_8.**

(a) -1 commutes with every element of Q_8, so $N \leq Z(Q_8)$ and is normal. The cosets are $N = \{1, -1\}$, $iN = \{i, -i\}$, $jN = \{j, -j\}$, and $kN = \{k, -k\}$. (12)

(b) Every nonidentity coset has square N, since $i^2 = j^2 = k^2 = -1 \in N$. Hence Q_8/N has four elements, all three nonidentity elements of order 2, and is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. A cyclic group of order 4 has an element of order 4, so this quotient is not cyclic. (18)

Grading note. Award normality credit for identifying the central subgroup. In (b), the three nonidentity cosets must all have order two.

Problem 3. (30 points) **A normalizer in S_4.**

(a) Any permutation commuting with $(1\,2\,3)$ must fix its unique fixed point 4 and restrict on $\{1,2,3\}$ to a power of that cycle. Thus $C_{S_4}(H) = H = \{1, (1\,2\,3), (1\,3\,2)\}$. (15)

(b) Conjugation sends $(1\,2\,3)$ to a generator of H precisely when it preserves its support $\{1,2,3\}$; all six permutations of that support do so. Hence $N_{S_4}(H) = S_{\{1,2,3\}} \cong S_3$, fixing 4. It is proper in S_4, so H is not normal. (15)

Grading note. In (a), distinguish commuting from merely preserving the subgroup. In (b), a normalizer of order six is not all of S_4.

Problem 4. (20 points) **The Second Isomorphism Theorem in D_{10}.**

(a) $N = \langle r \rangle$ has index 2 and is therefore normal. Since $H = \{1, s\}$ consists of reflections and the identity, while N consists of rotations, $H \cap N = 1$. The generators r, s belong to HN, so $HN = G$. (10)

(b) $H/(H \cap N) = H \cong \mathbb{Z}/2\mathbb{Z}$, and $HN/N = G/N$ has order 2, hence is also $\mathbb{Z}/2\mathbb{Z}$. The induced map $h(H \cap N) \mapsto hN$ sends s to its nontrivial coset and is an isomorphism. (10)

Grading note. In (a), check normality and both subgroup calculations. In (b), identify the two quotient groups and the induced map.