MATH 3551 A — Abstract Algebra I
Practice Midterm 2 — Version C — SOLUTIONS

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Problem 1. (20 points) **True or false; prove or disprove.**

(a) TRUE. For $k = \varphi(g)$ and $n = \varphi(x) \in \varphi(N)$, we have $knk^{-1} = \varphi(gxg^{-1}) \in \varphi(N)$ because $N \trianglelefteq G$.

(b) TRUE. If $G = \langle g \rangle$, every element of G/N is a power of gN, so $G/N = \langle gN \rangle$.

(c) FALSE. The symmetric group S_3 has order 6 and is not cyclic; no element has order 6.

(d) TRUE. The action homomorphism sends g to its permutation of the set; its kernel consists exactly of permutations acting as the identity, that is, fixing every point.

Grading note. Award full reason credit for a concrete noncyclic group of order 6, such as S_3, or for showing its elements have orders at most 3.

Problem 2. (30 points) **A matrix-group quotient.**

(a) The first column can be any nonzero vector of $\mathbb{F}_3^2$, giving $3^2 - 1 = 8$ choices. The second must lie outside the first column's one-dimensional span, leaving $3^2 - 3 = 6$ choices. Thus $|G| = 8 \cdot 6 = 48$. (12)

(b) For each $u \in \mathbb{F}_3^\times$, the diagonal matrix $\mathrm{diag}(u, 1)$ has determinant u, so det is onto. Its image has order 2, hence its kernel N has order $48/2 = 24$. The First Isomorphism Theorem gives $G/N \cong \mathbb{F}_3^\times \cong \mathbb{Z}/2\mathbb{Z}$. (18)

Grading note. In (a), count independent columns rather than all 3^4 matrices. In (b), exhibit determinants 1 and -1 and use the kernel-image order formula.

Problem 3. (30 points) **Subgroups and quotients of $\mathbb{Z}/18\mathbb{Z}$.**

(a) A cyclic group has exactly one subgroup for each divisor of its order. The subgroups are $\langle 0 \rangle$ (order 1), $\langle 9 \rangle$ (2), $\langle 6 \rangle$ (3), $\langle 3 \rangle$ (6), $\langle 2 \rangle$ (9), and $G = \langle 1 \rangle$ (18). (15)

(b) For $H = \langle 6 \rangle$ of order 3, the quotient has order 6 and is cyclic, hence $G/H \cong C_6$. For $H = \langle 3 \rangle$ of order 6, the quotient has order 3, hence $G/H \cong C_3$. (15)

Grading note. For the subgroup list, include the trivial subgroup and whole group. A quotient of a cyclic group is cyclic, and its order determines its isomorphism type.

Problem 4. (20 points) **A quotient of a quotient of Q_8.**

(a) $N = Z(Q_8)$ has order 2, so G/N has order 4; its three nonidentity cosets iN, jN, kN all have order 2, hence $G/N \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The group $K = \{1, -1, i, -i\}$ has order 4, so $K/N \cong \mathbb{Z}/2\mathbb{Z}$. Since K has index 2 in G, it is normal and $G/K \cong \mathbb{Z}/2\mathbb{Z}$. (10)

(b) Since $N \trianglelefteq K \trianglelefteq G$, the Third Isomorphism Theorem gives $(G/N)/(K/N) \cong G/K$ by $(gN)(K/N) \mapsto gK$. It is well defined because changing gN by an element of K/N changes g by an element of K; the map sends the two cosets of K/N to the two cosets of K. (10)

Grading note. Identify the Klein four quotient and both order-two quotients. In (b), give the induced map and its well-definedness.