MATH 3551 A — Abstract Algebra I
Practice Midterm 2 — SOLUTIONS

Taylor Dupuy, University of Vermont • Fall 2026

Fall 2026 • Total: 100 points

About this practice exam. It was drafted with the help of Claude Opus 5, running in a Cowork harness, working from previous exams in this course. **It has not yet been timed or proofread.** It is meant to run well inside a single class period, with time to spare. *Please hunt for mistakes.* If you find a wrong answer, an ambiguous or unanswerable problem, a problem that leans on something we have not covered, or one that takes far longer than its point value suggests, tell me — in class, in office hours, or by email. I would much rather hear it from you now than print it on the real exam.

Problem 1. (30 points) **True or false; prove or disprove.**

(a) **TRUE.** This is part of the Second Isomorphism Theorem: since $N \trianglelefteq G$ we have $HN = NH$, and a product of subgroups is a subgroup exactly when the two commute setwise.

(b) **FALSE.** Take $K = \langle (1\,2)(3\,4) \rangle$, $H = V_4$, $G = A_4$. Then $K \trianglelefteq H$ (H is abelian) and $H \trianglelefteq A_4$, but K is not normal in A_4: conjugating $(1\,2)(3\,4)$ by $(1\,2\,3)$ gives $(2\,3)(1\,4) \notin K$.

(c) **TRUE.** Conjugacy classes of S_n correspond to partitions of n, and 5 has seven partitions: 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1.

(d) **FALSE.** $\mathrm{Aut}(\mathbb{Z}/8\mathbb{Z}) \cong (\mathbb{Z}/8\mathbb{Z})^\times = \{\overline{1}, \overline{3}, \overline{5}, \overline{7}\}$, in which every element squares to $\overline{1}$; so it is the Klein four-group, not cyclic.

(e) **TRUE.** The class equation $|G| = |Z(G)| + \sum |G : C_G(g_i)|$ has every term of the sum divisible by p (each is an index > 1 of a subgroup of a p-group), and $p \mid |G|$; hence $p \mid |Z(G)|$, so $Z(G) \neq 1$.

(f) **TRUE.** $V_4 = \{1, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$ is a subgroup of order 4, and it is normal because it is a union of conjugacy classes of A_4 (the identity together with all elements of cycle type $(2,2)$, and conjugation preserves cycle type).

(g) **FALSE.** A product of 3-cycles is even, so it lies in A_n; a transposition is odd. The 3-cycles generate exactly A_n, not S_n.

(h) **TRUE.** Transitivity means A is a single orbit, so by orbit–stabilizer $|A| = |G : G_a|$ for any $a \in A$, and an index divides the order of a finite group by Lagrange.

(i) FALSE. The Klein four-group has order 2^2 and is not cyclic. (What *is* true is that every group of order p^2 is abelian.)

(j) TRUE. For $\sigma \in \mathrm{Aut}(G)$ and $\varphi_g \in \mathrm{Inn}(G)$ (the conjugation by g) one computes $\sigma\varphi_g\sigma^{-1} = \varphi_{\sigma(g)} \in \mathrm{Inn}(G)$.

Grading note. Items (b), (g) and (i) are where the points are lost. In (b) the counterexample must be explicit; "normality is not transitive" with no example is one reason-point. In (i) accept V_4 or $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Students who answer (e) by quoting "p-groups have nontrivial center" without saying which theorem produces it should get 1 of 2 reason-points — the class equation is the content.

Problem 2. (20 points) **An action of A_4.**

(a) For $(1\,2\,3)$: $\{1,2\} \mapsto \{2,3\}$ and $\{3,4\} \mapsto \{1,4\}$, so $P_1 \mapsto P_3$; $\{1,3\} \mapsto \{2,1\}$ and $\{2,4\} \mapsto \{3,4\}$, so $P_2 \mapsto P_1$; and hence $P_3 \mapsto P_2$. Thus

$$(1\,2\,3) \longmapsto (P_1\,P_3\,P_2).$$

For $(1\,2)(3\,4)$: $\{1,2\} \mapsto \{2,1\}$ and $\{3,4\} \mapsto \{4,3\}$, so $P_1 \mapsto P_1$; $\{1,3\} \mapsto \{2,4\}$ and $\{2,4\} \mapsto \{1,3\}$, so $P_2 \mapsto P_2$; similarly $P_3 \mapsto P_3$. Thus $(1\,2)(3\,4)$ acts as the *identity*.

(b) The same computation as above applies verbatim to $(1\,3)(2\,4)$ and $(1\,4)(2\,3)$ — each swaps the two entries within one splitting and swaps the two pairs of each of the others — so $V_4 \subseteq \ker\varphi$. Hence $|\operatorname{im}\varphi| = |A_4 : \ker\varphi| \le |A_4 : V_4| = 3$. On the other hand $\operatorname{im}\varphi$ contains the 3-cycle $(P_1\,P_3\,P_2)$, so $|\operatorname{im}\varphi| \ge 3$. Therefore

$$\ker\varphi = V_4 \quad \text{and} \quad \operatorname{im}\varphi = \langle (P_1\,P_3\,P_2)\rangle \cong \mathbb{Z}/3\mathbb{Z}.$$

In particular φ is *not* surjective onto S_3.

(c) A kernel is always normal, so $V_4 \trianglelefteq A_4$; and by the First Isomorphism Theorem $A_4/V_4 \cong \operatorname{im}\varphi \cong \mathbb{Z}/3\mathbb{Z}$.

(d) By (a) and (b), V_4 fixes P_1, and $|V_4| = 4$. The orbit of P_1 is all of A (the 3-cycle above moves it), so orbit–stabilizer gives $|(A_4)_{P_1}| = 12/3 = 4$; hence $(A_4)_{P_1} = V_4$ exactly. Check: $4 \cdot 3 = 12 = |A_4|$.

Grading note. The intended surprise is that the representation is *not* surjective — a student who assumes $\varphi\colon A_4 \to S_3$ must be onto because $|A_4| > |S_3|$ has made the error this problem exists to catch. In (b), award 3 points for $V_4 \subseteq \ker\varphi$ and 3 for the order argument pinning down equality. In (d) accept the answer V_4 obtained either by direct check or by orbit–stabilizer.

Problem 3. (15 points) **Conjugacy in S_7.**

(a) An element of order 4 has cycle lengths with lcm 4, so its cycle type (as a partition of 7) must contain a 4 and no 3: the possibilities are $(4,1,1,1)$ and $(4,2,1)$. Counting,

$$\#(4,1,1,1) = \frac{7!}{4 \cdot 1^3 \cdot 3!} = \frac{5040}{24} = 210, \qquad \#(4,2,1) = \frac{7!}{4 \cdot 2 \cdot 1} = \frac{5040}{8} = 630.$$

Total: $210 + 630 = \boxed{840}$.

(b) $(1\,2\,3\,4)(5\,6)$ has cycle type $(4,2,1)$, so its class has 630 elements.

(c) $|C_{S_7}(\sigma)| = 7!/630 = 8$ by orbit–stabilizer for the conjugation action. The eight elements are the products $(1\,2\,3\,4)^a(5\,6)^b$ with $0 \le a < 4$, $0 \le b < 2$:

$$1,\ (1\,2\,3\,4),\ (1\,3)(2\,4),\ (1\,4\,3\,2),\ (5\,6),\ (1\,2\,3\,4)(5\,6),\ (1\,3)(2\,4)(5\,6),\ (1\,4\,3\,2)(5\,6),$$

an abelian group isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Grading note. The denominator 3! in the count of type $(4, 1, 1, 1)$ — the reordering of the three fixed points — is the standard place to lose points; a student who writes $7!/4 = 1260$ has forgotten it. Award 3 of 6 for identifying both cycle types with one count wrong. In (c) accept "the group generated by $(1\,2\,3\,4)$ and $(5\,6)$" as the explicit description.

Problem 4. (20 points) **Actions on cosets.**

(a) If $g \in \ker \pi$ then g fixes every coset; in particular $g \cdot (1 \cdot H) = 1 \cdot H$, i.e. $gH = H$, i.e. $g \in H$. Hence $\ker \pi \leq H$.

(b) Let $K \trianglelefteq G$ with $K \leq H$, and let $k \in K$. For any $x \in G$ we have $x^{-1}kx \in K \leq H$ (normality), so $kxH = x(x^{-1}kx)H = xH$. Thus k fixes every coset, i.e. $k \in \ker \pi$. Hence $K \leq \ker \pi$.

For the index: by the First Isomorphism Theorem, $G/\ker \pi \cong \operatorname{im} \pi \leq S_n$, so $|G : \ker \pi| = |\operatorname{im} \pi|$ divides $|S_n| = n!$ by Lagrange.

(c) Here $n = 4$, so $|G : \ker \pi|$ divides $4! = 24$. Consequently

$$|\ker \pi| \;=\; \frac{48}{|G : \ker \pi|} \;\geq\; \frac{48}{24} \;=\; 2,$$

so $\ker \pi \neq 1$. Also $\ker \pi \leq H$ and $|G : H| = 4 > 1$, so $\ker \pi \neq G$. Thus $\ker \pi$ is a nontrivial proper normal subgroup of G, and G is not simple.

Grading note. This is D&F §4.2 Exercise 8 and it is the engine behind most "G is not simple" arguments; students who have done it should finish in five minutes. In (a) the whole content is evaluating the action at the coset H itself. In (c) *both* halves must appear — $\ker \pi \neq 1$ from the order count, and $\ker \pi \neq G$ from $\ker \pi \leq H \lneq G$; a solution producing only one of the two is worth 3 of the 6 points. Expected error: concluding $|G : \ker \pi|$ divides $n = 4$ rather than $n! = 24$, which happens to give the same conclusion here but is wrong.

Problem 5. (15 points) **Composition series.**

(a) A *composition series* for G is a chain of subgroups $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_k = G$ in which each N_i is normal in N_{i+1} and each quotient N_{i+1}/N_i is simple. The quotients N_{i+1}/N_i are the *composition factors*.

(b) Take

$$D_{12} \;\rhd\; \langle r \rangle \;\rhd\; \langle r^2 \rangle \;\rhd\; 1.$$

Each containment is of prime index (2, 2, 3), hence normal with simple (cyclic of prime order) quotient. Reading from the top, the composition factors are

$$D_{12}/\langle r \rangle \cong \mathbb{Z}/2\mathbb{Z}, \qquad \langle r \rangle/\langle r^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}, \qquad \langle r^2 \rangle \cong \mathbb{Z}/3\mathbb{Z}.$$

(c) Take

$$D_{12} \;\rhd\; \langle r \rangle \;\rhd\; \langle r^3 \rangle \;\rhd\; 1,$$

where $\langle r^3 \rangle$ has order 2. The factors, from the top, are $\mathbb{Z}/2\mathbb{Z}$, then $\langle r \rangle/\langle r^3 \rangle \cong \mathbb{Z}/3\mathbb{Z}$, then $\langle r^3 \rangle \cong \mathbb{Z}/2\mathbb{Z}$ — the same three groups in a different order. The Jordan–Hölder Theorem predicts exactly this: any two composition series of a finite group have the same length and the same multiset of composition factors up to isomorphism and reordering. Here both lists are $\{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}\}$.

(Another genuinely different series is $D_{12} \rhd \langle r^2, s \rangle \rhd \langle r^2 \rangle \rhd 1$, with $\langle r^2, s \rangle \cong S_3$.)

(d) Yes. All the composition factors are cyclic of prime order, hence abelian, so the series in (b) is an abelian series and D_{12} is solvable.

Grading note. In (a) the two things that must appear are "each term normal in the next" (not normal in G) and "successive quotients simple". In (b) and (c) the essential check is that each index is prime; students who write a chain without justifying simplicity of the factors lose 2 points. The commonest wrong answer in (c) is to give $D_{12} \rhd \langle r^2, s\rangle \rhd \langle r^2\rangle \rhd 1$, which is a different series but produces the factors in the *same* order $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$; award 3 of 5 — it is a correct second series but does not answer the question asked.