MATH 3551 A — Abstract Algebra I
Practice Midterm 2 — Version 2 — SOLUTIONS

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What changed from Version 1. The one-week coverage freeze moves §4.4 to Midterm 3. The automorphism true/false item was replaced by a group-action item; every other question is unchanged. Material deferred from Midterm 1 — §1.7, §2.2, and §§3.1–3.2 — is included in this coverage window. Version 1 remains available on the course webpage so that the changes can be compared directly.

This practice exam targets about thirty minutes of work in a fifty-minute period. Please report any error, ambiguity, out-of-coverage question, or item that takes much longer than its point value suggests.

Problem 1. (30 points) **True or false; prove or disprove.**

(a) **TRUE.** This is part of the Second Isomorphism Theorem: since $N \trianglelefteq G$ we have $HN = NH$, and a product of subgroups is a subgroup exactly when the two commute setwise.

(b) **FALSE.** Take $K = \langle (1\,2)(3\,4) \rangle$, $H = V_4$, $G = A_4$. Then $K \trianglelefteq H$ (H is abelian) and $H \trianglelefteq A_4$, but K is not normal in A_4: conjugating $(1\,2)(3\,4)$ by $(1\,2\,3)$ gives $(2\,3)(1\,4) \notin K$.

(c) **TRUE.** Conjugacy classes of S_n correspond to partitions of n, and 5 has seven partitions: 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1.

(d) **TRUE.** A faithful action gives an injective homomorphism $G \hookrightarrow S_5$. Thus G is isomorphic to a subgroup of S_5, and Lagrange's Theorem gives $|G| \mid |S_5| = 5! = 120$.

(e) **TRUE.** The class equation $|G| = |Z(G)| + \sum |G : C_G(g_i)|$ has every term of the sum divisible by p (each is an index > 1 of a subgroup of a p-group), and $p \mid |G|$; hence $p \mid |Z(G)|$, so $Z(G) \neq 1$.

(f) **TRUE.** $V_4 = \{1, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$ is a subgroup of order 4, and it is normal because it is a union of conjugacy classes of A_4 (the identity together with all elements of cycle type $(2, 2)$, and conjugation preserves cycle type).

Grading note. Most of the credit here is in the *reason*, and the reason must be specific: a named counterexample, the actual permutation, or the actual count. A circled answer with no reason earns two of the five points; a wrong circle with a reason that is right about the underlying fact earns the three reason-points. Give full credit for a correct counterexample even if it is not the one printed above. The parenthetical remarks in each answer flag the error to expect on that item.

Problem 2. (25 points) **An action of A_4.**

(a) For $(1\,2\,3)$: $\{1,2\} \mapsto \{2,3\}$ and $\{3,4\} \mapsto \{1,4\}$, so $P_1 \mapsto P_3$; $\{1,3\} \mapsto \{2,1\}$ and $\{2,4\} \mapsto \{3,4\}$, so $P_2 \mapsto P_1$; and hence $P_3 \mapsto P_2$. Thus

$$(1\,2\,3) \longmapsto (P_1\,P_3\,P_2).$$

For $(1\,2)(3\,4)$: $\{1,2\} \mapsto \{2,1\}$ and $\{3,4\} \mapsto \{4,3\}$, so $P_1 \mapsto P_1$; $\{1,3\} \mapsto \{2,4\}$ and $\{2,4\} \mapsto \{1,3\}$, so $P_2 \mapsto P_2$; similarly $P_3 \mapsto P_3$. Thus $(1\,2)(3\,4)$ acts as the *identity*.

(b) The same computation as above applies verbatim to $(1\,3)(2\,4)$ and $(1\,4)(2\,3)$ — each swaps the two entries within one splitting and swaps the two pairs of each of the others — so $V_4 \subseteq \ker\varphi$. Hence $|\operatorname{im}\varphi| = |A_4 : \ker\varphi| \le |A_4 : V_4| = 3$. On the other hand $\operatorname{im}\varphi$ contains the 3-cycle $(P_1\,P_3\,P_2)$, so $|\operatorname{im}\varphi| \ge 3$. Therefore

$$\ker\varphi = V_4 \quad\text{and}\quad \operatorname{im}\varphi = \langle (P_1\,P_3\,P_2)\rangle \cong \mathbb{Z}/3\mathbb{Z}.$$

In particular φ is *not* surjective onto S_3.

(c) A kernel is always normal, so $V_4 \trianglelefteq A_4$; and by the First Isomorphism Theorem $A_4/V_4 \cong \operatorname{im}\varphi \cong \mathbb{Z}/3\mathbb{Z}$.

(d) By (a) and (b), V_4 fixes P_1, and $|V_4| = 4$. The orbit of P_1 is all of A (the 3-cycle above moves it), so orbit–stabilizer gives $|(A_4)_{P_1}| = 12/3 = 4$; hence $(A_4)_{P_1} = V_4$ exactly. Check: $4 \cdot 3 = 12 = |A_4|$.

Grading note. The intended surprise is that the representation is *not* surjective — a student who assumes $\varphi\colon A_4 \to S_3$ must be onto because $|A_4| > |S_3|$ has made the error this problem exists to catch. In (b), award 3 points for $V_4 \subseteq \ker\varphi$ and 3 for the order argument pinning down equality. In (d) accept the answer V_4 obtained either by direct check or by orbit–stabilizer.

Problem 3. (25 points) **Conjugacy in S_7.**

(a) An element of order 4 has cycle lengths with lcm 4, so its cycle type (as a partition of 7) must contain a 4 and no 3: the possibilities are $(4,1,1,1)$ and $(4,2,1)$. Counting,

$$\#(4,1,1,1) = \frac{7!}{4 \cdot 1^3 \cdot 3!} = \frac{5040}{24} = 210, \qquad \#(4,2,1) = \frac{7!}{4 \cdot 2 \cdot 1} = \frac{5040}{8} = 630.$$

Total: $210 + 630 = \boxed{840}$.

(b) $(1\,2\,3\,4)(5\,6)$ has cycle type $(4,2,1)$, so its class has 630 elements.

(c) $|C_{S_7}(\sigma)| = 7!/630 = 8$ by orbit–stabilizer for the conjugation action. The eight elements are the products $(1\,2\,3\,4)^a(5\,6)^b$ with $0 \le a < 4$, $0 \le b < 2$:

$$1,\ (1\,2\,3\,4),\ (1\,3)(2\,4),\ (1\,4\,3\,2),\ (5\,6),\ (1\,2\,3\,4)(5\,6),\ (1\,3)(2\,4)(5\,6),\ (1\,4\,3\,2)(5\,6),$$

an abelian group isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Grading note. The denominator $3!$ in the count of type $(4, 1, 1, 1)$ — the reordering of the three fixed points — is the standard place to lose points; a student who writes $7!/4 = 1260$ has forgotten it. Award 3 of 6 for identifying both cycle types with one count wrong. In (c) accept "the group generated by $(1\,2\,3\,4)$ and $(5\,6)$" as the explicit description.

Problem 4. (20 points) **Composition series.**

(a) A *composition series* for G is a chain of subgroups $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_k = G$ in which each N_i is normal in N_{i+1} and each quotient N_{i+1}/N_i is simple. The quotients N_{i+1}/N_i are the *composition factors*.

(b) Take

$$D_{12} \;\triangleright\; \langle r\rangle \;\triangleright\; \langle r^2\rangle \;\triangleright\; 1.$$

Each containment is of prime index (2, 2, 3), hence normal with simple (cyclic of prime order) quotient. Reading from the top, the composition factors are

$$D_{12}/\langle r\rangle \cong \mathbb{Z}/2\mathbb{Z}, \qquad \langle r\rangle/\langle r^2\rangle \cong \mathbb{Z}/2\mathbb{Z}, \qquad \langle r^2\rangle \cong \mathbb{Z}/3\mathbb{Z}.$$

(c) Take

$$D_{12} \;\triangleright\; \langle r\rangle \;\triangleright\; \langle r^3\rangle \;\triangleright\; 1,$$

where $\langle r^3\rangle$ has order 2. The factors, from the top, are $\mathbb{Z}/2\mathbb{Z}$, then $\langle r\rangle/\langle r^3\rangle \cong \mathbb{Z}/3\mathbb{Z}$, then $\langle r^3\rangle \cong \mathbb{Z}/2\mathbb{Z}$ — the same three groups in a different order. The Jordan–Hölder Theorem predicts exactly this: any two composition series of a finite group have the same length and the same multiset of composition factors up to isomorphism and reordering. Here both lists are $\{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}\}$.

(Another genuinely different series is $D_{12} \triangleright \langle r^2, s\rangle \triangleright \langle r^2\rangle \triangleright 1$, with $\langle r^2, s\rangle \cong S_3$.)

(d) Yes. All the composition factors are cyclic of prime order, hence abelian, so the series in (b) is an abelian series and D_{12} is solvable.

Grading note. In (a) the two things that must appear are "each term normal in the next" (not normal in G) and "successive quotients simple". In (b) and (c) the essential check is that each index is prime; students who write a chain without justifying simplicity of the factors lose 2 points. The commonest wrong answer in (c) is to give $D_{12} \triangleright \langle r^2, s\rangle \triangleright \langle r^2\rangle \triangleright 1$, which is a different series but produces the factors in the *same* order $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$; award 3 of 5 — it is a correct second series but does not answer the question asked.