MATH 3551 A — Abstract Algebra I
Practice Midterm 2

Taylor Dupuy, University of Vermont • Fall 2026

For practice — Midterm 2 is Friday, November 6, 2026 • Time limit: 50 minutes • Total: 100 points

---

NAME (please print): ____

Directions. This exam is **closed book**: no textbook, no notes, no calculators, no phones, no other devices, and no assistance of any kind. Write your answers on the exam itself; use the backs of pages if you need more room, and say clearly when you have done so. *Show your work and name the theorems you use* — "by Lagrange's Theorem, $|H|$ divides 24" earns nearly all of the credit, an unjustified correct number earns little. Point values are marked on each problem. Good luck.

About this practice exam. It was drafted with the help of Claude Opus 5, running in a Cowork harness, working from previous exams in this course. **It has not yet been timed or proofread.** It is meant to run well inside a single class period, with time to spare. *Please hunt for mistakes.* If you find a wrong answer, an ambiguous or unanswerable problem, a problem that leans on something we have not covered, or one that takes far longer than its point value suggests, tell me — in class, in office hours, or by email. I would much rather hear it from you now than print it on the real exam.

Problem 1. (30 points) **True or false; prove or disprove.**

Circle **T** or **F**, and then give a one-line reason: a theorem, a short computation, or an explicit counterexample. Three points each.

(a) **T** **F** If $H \leq G$ and $N \trianglelefteq G$, then HN is a subgroup of G.

(b) **T** **F** If $K \trianglelefteq H$ and $H \trianglelefteq G$, then $K \trianglelefteq G$.

(c) **T** **F** The symmetric group S_5 has exactly seven conjugacy classes.

(d) **T** **F** The automorphism group $\mathrm{Aut}(\mathbb{Z}/8\mathbb{Z})$ is cyclic.

(e) **T** **F** If $|G| = p^3$ for a prime p, then $Z(G) \neq 1$.

(f) **T** **F** The alternating group A_4 has a normal subgroup of order 4.

(g) **T** **F** For $n \geq 3$, every permutation in S_n is a product of 3-cycles.

(h) **T** **F** If a finite group G acts transitively on a set A, then $|A|$ divides $|G|$.

(i) **T** **F** Every group of order p^2, p prime, is cyclic.

(j) **T** **F** $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

Problem 2. (20 points) **An action of A_4.**

Let $G = A_4$ act on the set $A = \{P_1, P_2, P_3\}$ of the three ways of splitting $\{1, 2, 3, 4\}$ into two unordered pairs:

$$P_1 = \{\{1,2\},\{3,4\}\}, \quad P_2 = \{\{1,3\},\{2,4\}\}, \quad P_3 = \{\{1,4\},\{2,3\}\},$$

where σ acts on a pair by $\sigma \cdot \{i, j\} = \{\sigma(i), \sigma(j)\}$ and on a splitting by acting on each of its two pairs.

(a) (6 points) Determine the permutations of A induced by $(1\,2\,3)$ and by $(1\,2)(3\,4)$.

(b) (6 points) Let $\varphi\colon A_4 \to S_3$ be the resulting permutation representation. Determine $\ker\varphi$ and $\operatorname{im}\varphi$, with proof.

(c) (4 points) Deduce that $V_4 = \{1, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$ is a normal subgroup of A_4 and identify the quotient A_4/V_4.

(d) (4 points) Determine the stabilizer of P_1 in A_4 and verify the orbit–stabilizer relation.

Problem 3. (15 points) **Conjugacy in S_7.**

(a) (6 points) Determine the number of elements of order 4 in S_7.

(b) (5 points) Determine the size of the conjugacy class of $(1\,2\,3\,4)(5\,6)$ in S_7.

(c) (4 points) Determine the order of $C_{S_7}\big((1\,2\,3\,4)(5\,6)\big)$ and describe its elements explicitly.

Problem 4. (20 points) **Actions on cosets.**

Let G be a finite group and let $H \leq G$ with $|G : H| = n$. Let G act on the set of n left cosets of H in G by left multiplication, $g \cdot (xH) = gxH$, and let $\pi \colon G \to S_n$ be the resulting permutation representation.

(a) (6 points) Prove that $\ker \pi \leq H$.

(b) (8 points) Prove that $\ker \pi$ contains every normal subgroup of G that is contained in H, and that $|G : \ker \pi|$ divides $n!$.

(c) (6 points) Suppose now that $|G| = 48$ and that G has a subgroup H with $|G : H| = 4$. Prove that G is not simple.

Problem 5. (15 points) **Composition series.**

(a) (3 points) Define *composition series* and *composition factors*.

(b) (5 points) Give a composition series for $D_{12} = \langle r, s \mid r^6 = s^2 = 1,\ rs = sr^{-1} \rangle$ and list its composition factors, in order.

(c) (5 points) Give a second composition series for D_{12}, different from the one in (b), in which the composition factors appear in a *different order*. What does the Jordan–Hölder Theorem predict about the two lists?

(d) (2 points) Is D_{12} solvable? Justify your answer from (b).

End of practice exam. Total: $30 + 20 + 15 + 20 + 15 = 100$ points.