MATH 3551 A — Abstract Algebra I
Practice Midterm 3 — SOLUTIONS

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About this practice exam. It was drafted with the help of Claude Opus 5, running in a Cowork harness, working from previous exams in this course. **It has not yet been timed or proofread.** It is meant to run well inside a single class period, with time to spare. *Please hunt for mistakes.* If you find a wrong answer, an ambiguous or unanswerable problem, a problem that leans on something we have not covered, or one that takes far longer than its point value suggests, tell me — in class, in office hours, or by email. I would much rather hear it from you now than print it on the real exam.

Problem 1. (30 points) **True or false; prove or disprove.**

(a) **TRUE.** $35 = 5 \cdot 7$: $n_7 \mid 5$ and $n_7 \equiv 1 \pmod 7$ force $n_7 = 1$; $n_5 \mid 7$ and $n_5 \equiv 1 \pmod 5$ force $n_5 = 1$. Both Sylow subgroups are normal of coprime prime order, so $G \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \cong \mathbb{Z}/35\mathbb{Z}$.

(b) **FALSE.** $351 = 3^3 \cdot 13$, so $n_{13} \in \{1, 27\}$. If $n_{13} = 27$ there are $27 \cdot 12 = 324$ elements of order 13, leaving $351 - 324 = 27$ elements — exactly enough for a single Sylow 3-subgroup, so $n_3 = 1$. Either way G has a normal Sylow subgroup.

(c) **FALSE.** $V_4 = \{1, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$ is a normal subgroup of A_4 of order 4.

(d) **FALSE.** The number of abelian groups of order p^5 is the number of partitions of 5, which is 7: 5; 4+1; 3+2; 3+1+1; 2+2+1; 2+1+1+1; 1+1+1+1+1.

(e) **TRUE.** $\mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/18\mathbb{Z}$ has elementary divisors $4, 2$ (at $p = 2$) and $3, 9$ (at $p = 3$); $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/36\mathbb{Z}$ has $2, 4$ and $3, 9$. The lists agree, so the groups are isomorphic. (Both have invariant factors $6 \mid 36$.)

(f) **TRUE.** By Sylow's Theorem any two Sylow p-subgroups are conjugate, and conjugate subgroups are isomorphic via $x \mapsto gxg^{-1}$.

(g) **TRUE.** $|S_4| = 24 = 2^3 \cdot 3$, so $n_2 \mid 3$ and n_2 is odd, giving $n_2 \in \{1, 3\}$. A Sylow 2-subgroup has order 8; the three copies of D_8 stabilising the three ways of pairing up the four points are distinct, so $n_2 = 3$.

(h) **FALSE.** The dihedral group D_{100} has order 100 and is nonabelian ($rs \neq sr$).

(i) TRUE. This is the standard consequence of the Fundamental Theorem: writing $A \cong \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_t\mathbb{Z}$ with $n_1 \mid \cdots \mid n_t$, the order of any element divides n_t (so n_t is the exponent of A), and the last factor supplies an element of order m for each $m \mid n_t$.

(j) FALSE. A subgroup isomorphic to $\mathbb{Z}/6\mathbb{Z}$ would contain an element of order 6. But every element of A_5 has cycle type 1^5, $(2, 2, 1)$, $(3, 1, 1)$ or (5), so its order is 1, 2, 3 or 5.

Grading note. The Sylow items (a), (b), (g) need the actual congruence, not just the phrase "by Sylow". In (e) accept either the elementary-divisor comparison or the invariant factors $6 \mid 36$; students who say "both have order 216" have not answered. Item (j) is the one that rewards knowing the element orders in A_5 cold.

Problem 2. (20 points) **A group of order 195.**

(a) $|G| = 3 \cdot 5 \cdot 13$.

- $n_{13} \mid 15$ and $n_{13} \equiv 1 \pmod{13}$: the divisors $1, 3, 5, 15$ reduce to $1, 3, 5, 2$ mod 13, so $n_{13} = 1$.
- $n_5 \mid 39$ and $n_5 \equiv 1 \pmod 5$: the divisors $1, 3, 13, 39$ reduce to $1, 3, 3, 4$ mod 5, so $n_5 = 1$.
- $n_3 \mid 65$ and $n_3 \equiv 1 \pmod 3$: the divisors $1, 5, 13, 65$ reduce to $1, 2, 1, 2$ mod 3, so $n_3 \in \{1, 13\}$.

(b) Let $P \in \mathrm{Syl}_5(G)$ and $R \in \mathrm{Syl}_{13}(G)$; by (a) both are normal in G. Then PR is a subgroup (a product of subgroups is a subgroup as soon as one of them is normal), and since $|P \cap R|$ divides $\gcd(5, 13) = 1$,

$$|PR| = \frac{|P|\,|R|}{|P \cap R|} = 5 \cdot 13 = 65.$$

It is normal: for any $g \in G$, $g(PR)g^{-1} = (gPg^{-1})(gRg^{-1}) = PR$. So $H := PR \trianglelefteq G$ has order 65.

(c) Consider

$$1 \ \trianglelefteq\ R \ \trianglelefteq\ H \ \trianglelefteq\ G.$$

Each containment is legitimate: $R \trianglelefteq G$ so certainly $R \trianglelefteq H$, and $H \trianglelefteq G$ by (b). The successive quotients are

$$R \cong \mathbb{Z}/13\mathbb{Z}, \qquad H/R \text{ of order } 5 \cong \mathbb{Z}/5\mathbb{Z}, \qquad G/H \text{ of order } 3 \cong \mathbb{Z}/3\mathbb{Z},$$

all abelian (indeed cyclic of prime order). Hence G is solvable.

Grading note. In (a) award 2 points per prime, and only for a stated congruence — the answer $n_{13} = 1$ with no reduction of 15 mod 13 is worth 1. In (b) the two things to check are that PR is a *subgroup* (normality of one factor) and that $|PR| = 65$ (trivial intersection); a solution that asserts $|PR| = 65$ without the intersection argument gets 5 of 8. In (c) it is enough to exhibit a chain with abelian quotients; students need not verify simplicity of the factors, since solvability does not require it.

Problem 3. (20 points) **Invariant factors and elementary divisors.**

(a) $20 = 2^2 \cdot 5$, $30 = 2 \cdot 3 \cdot 5$, $50 = 2 \cdot 5^2$, so

$$|A| = 20 \cdot 30 \cdot 50 = 30000 = 2^4 \cdot 3 \cdot 5^4.$$

Splitting each cyclic factor into its prime-power parts gives the elementary divisors

$$\underbrace{4,\ 2,\ 2,}_{p=2} \qquad \underbrace{3}_{p=3}, \qquad \underbrace{25,\ 5,\ 5}_{p=5}.$$

The longest of these lists has three entries, so there are three invariant factors. Align the lists from the largest, padding with 1s, and multiply down the columns:

	col 1	col 2	col 3
$p = 2$	4	2	2
$p = 3$	3	1	1
$p = 5$	25	5	5
	300	10	10

so the invariant factors are $10 \mid 10 \mid 300$, i.e. $A \cong \mathbb{Z}_{10} \times \mathbb{Z}_{10} \times \mathbb{Z}_{300}$. (Check: $10 \cdot 10 \cdot 300 = 30000$.)

(b) The number of abelian groups of order $\prod p_i^{a_i}$ is $\prod p(a_i)$, where $p(a)$ is the number of partitions of a. Here $30000 = 2^4 \cdot 3^1 \cdot 5^4$, so the count is $p(4) \cdot p(1) \cdot p(4) = 5 \cdot 1 \cdot 5 = \boxed{25}$.

(c) $392 = 2^3 \cdot 7^2$. The partitions of 3 give 2-parts (8), $(4, 2)$, $(2, 2, 2)$; the partitions of 2 give 7-parts (49), $(7, 7)$. That is $3 \cdot 2 = 6$ groups:

elementary divisors	invariant factors
$8;\ 49$	392
$8;\ 7, 7$	$7 \mid 56$
$4, 2;\ 49$	$2 \mid 196$
$4, 2;\ 7, 7$	$14 \mid 28$
$2, 2, 2;\ 49$	$2 \mid 2 \mid 98$
$2, 2, 2;\ 7, 7$	$2 \mid 14 \mid 14$

Grading note. Part (a): 2 points for the factorisation of the order, 3 for the elementary divisors, 3 for the invariant factors. The single most common error is to align the prime lists from the *left* without padding, which produces $4 \cdot 3 \cdot 25 = 300$ in the wrong column or an incorrect number of factors; insist on the "align from the largest" rule. In (c) each line is worth 1.33 points; a student who lists elementary divisors only and does not convert should get 4 of the 8.

Problem 4. (15 points) **A direct product.**

Throughout, $Z(G_1 \times G_2) = Z(G_1) \times Z(G_2)$, and a Sylow p-subgroup of $G_1 \times G_2$ is exactly a product $P_1 \times P_2$ of Sylow p-subgroups of the factors — so $n_p(G_1 \times G_2) = n_p(G_1)\, n_p(G_2)$.

(a) $|G| = 12 \cdot 15 = 180 = 2^2 \cdot 3^2 \cdot 5$. Since $Z(A_4) = 1$ (no nonidentity element of A_4 commutes with everything: a 3-cycle is moved by conjugation, and a double transposition is moved by a 3-cycle) and $\mathbb{Z}/15\mathbb{Z}$ is abelian,

$$Z(G) = Z(A_4) \times Z(\mathbb{Z}/15\mathbb{Z}) = 1 \times \mathbb{Z}/15\mathbb{Z} \cong \mathbb{Z}/15\mathbb{Z}.$$

(b) A Sylow 3-subgroup of G has order 9 and is $P_1 \times P_2$ with $P_1 \in \mathrm{Syl}_3(A_4)$ (order 3) and $P_2 \in \mathrm{Syl}_3(\mathbb{Z}/15\mathbb{Z})$ (order 3). Now $n_3(A_4) = 4$ — the four subgroups generated by 3-cycles — and $n_3(\mathbb{Z}/15\mathbb{Z}) = 1$ since $\mathbb{Z}/15\mathbb{Z}$ is abelian. Hence $n_3(G) = 4 \cdot 1 = \boxed{4}$.

(c) The order of $(a, b) \in A_4 \times \mathbb{Z}/15\mathbb{Z}$ is $\mathrm{lcm}(|a|, |b|)$. The element orders in A_4 are $1, 2, 3$; those in $\mathbb{Z}/15\mathbb{Z}$ are $1, 3, 5, 15$. The largest lcm available is $\mathrm{lcm}(2, 15) = 30$, attained by e.g. $((1\,2)(3\,4), \bar{1})$. So the answer is $\boxed{30}$. (Note 180 is not attained: G is not cyclic.)

(d) A Sylow 2-subgroup of G has order 4 and equals $V_4 \times 1$, since V_4 is the unique Sylow 2-subgroup of A_4 ($V_4 \trianglelefteq A_4$) and $\mathbb{Z}/15\mathbb{Z}$ has odd order. Hence $n_2(G) = 1 \cdot 1 = \boxed{1}$.

Grading note. The two facts being tested are D&F §5.1 #1 and #4, and students who know them finish in four minutes. The expected errors: in (a), writing $Z(G) = 1$ because A_4 has trivial centre — the centre of a product is a product, not an intersection; in (c), answering 180 (assuming G is cyclic because gcds look coprime) or 15 (forgetting to take the lcm with an element of order 2). Award 2 of 4 in (b) for the correct value of $n_3(A_4)$ alone.

Problem 5. (15 points) **The conjugacy classes of A_5.**

(a) The elements of A_5 are the even permutations of $\{1, \ldots, 5\}$, whose cycle types and counts are

$$\begin{array}{lll} 1^5 & : & 1 \text{ element} \\ (2,2,1) & : & \dfrac{5!}{2^2 \cdot 2!} = 15 \text{ elements} \\ (3,1,1) & : & \dfrac{5!}{3 \cdot 2!} = 20 \text{ elements} \\ (5) & : & 4! = 24 \text{ elements} \end{array}$$

Total $1 + 15 + 20 + 24 = 60 = |A_5|$. ✓

(b) An S_5-class contained in A_5 either stays a single A_5-class or splits into two of equal size, according to whether the S_5-centralizer of a representative contains an odd permutation. Compute:

- $\sigma = (1\,2)(3\,4)$: $|C_{S_5}(\sigma)| = 120/15 = 8$, and $(1\,2) \in C_{S_5}(\sigma)$ is odd, so $|C_{A_5}(\sigma)| = 4$ and the class has $60/4 = 15$ elements — it does *not* split.

- $\sigma = (1\,2\,3)$: $|C_{S_5}(\sigma)| = 120/20 = 6$, and $(4\,5) \in C_{S_5}(\sigma)$ is odd, so $|C_{A_5}(\sigma)| = 3$ and the class has $60/3 = 20$ elements — it does not split.
- $\sigma = (1\,2\,3\,4\,5)$: $|C_{S_5}(\sigma)| = 120/24 = 5$, so $C_{S_5}(\sigma) = \langle\sigma\rangle$, all of whose elements are even. Hence $|C_{A_5}(\sigma)| = 5$ and each A_5-class of 5-cycles has $60/5 = 12$ elements: the 24 five-cycles split into *two* classes of 12.

So the class sizes are $1,\ 15,\ 20,\ 12,\ 12$ (summing to 60).

(c) Suppose $N \trianglelefteq A_5$. Then N is a union of conjugacy classes, one of which is $\{1\}$, and $|N|$ divides 60 by Lagrange. So $|N|$ is 1 plus a sub-sum of $\{15, 20, 12, 12\}$. The possible values are

$$1,\ 13,\ 16,\ 21,\ 25,\ 28,\ 33,\ 36,\ 40,\ 45,\ 48,\ 60,$$

and of these only 1 and 60 divide 60. Hence $N = 1$ or $N = A_5$, i.e. A_5 is simple.

Grading note. Part (b) is the discriminating part: the answer $1, 15, 20, 24$ (that is, forgetting the split of the 5-cycles) is the expected error and should cost 3 of the 5 points, and it also breaks (c), since $1+24 = 25$ still fails to divide 60 — so a student can reach the right conclusion from the wrong class list. Grade (b) and (c) independently. In (c) the essential point is "normal subgroup = union of classes containing 1, and its order divides 60"; an exhaustive list of sub-sums is not required if the student explains why no sub-sum works.