MATH 3551 A — Abstract Algebra I
Practice Midterm 3 — Version 2 — SOLUTIONS

Taylor Dupuy, University of Vermont • Fall 2026

Fall 2026 • Total: 100 points

Version 2 — September 28, 2026. Generated by GPT-5.6 Sol in the Codex desktop harness. Please notify Professor Dupuy if something looks off.

What changed from Version 1. The one-week coverage freeze moves §4.4 from Midterm 2 to Midterm 3. One true/false item was replaced by an automorphism item; every other question is unchanged. Version 1 remains available on the course webpage so that the changes can be compared directly.

This practice exam targets about thirty minutes of work in a fifty-minute period. Please report any error, ambiguity, out-of-coverage question, or item that takes much longer than its point value suggests.

Problem 1. (30 points) **True or false; prove or disprove.**

(a) **TRUE.** $35 = 5 \cdot 7$: $n_7 \mid 5$ and $n_7 \equiv 1 \pmod 7$ force $n_7 = 1$; $n_5 \mid 7$ and $n_5 \equiv 1 \pmod 5$ force $n_5 = 1$. Both Sylow subgroups are normal of coprime prime order, so $G \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \cong \mathbb{Z}/35\mathbb{Z}$.

(b) **FALSE.** $351 = 3^3 \cdot 13$, so $n_{13} \in \{1, 27\}$. If $n_{13} = 27$ there are $27 \cdot 12 = 324$ elements of order 13, leaving $351 - 324 = 27$ elements — exactly enough for a single Sylow 3-subgroup, so $n_3 = 1$. Either way G has a normal Sylow subgroup.

(c) **FALSE.** The number of abelian groups of order p^5 is the number of partitions of 5, which is 7: 5; 4+1; 3+2; 3+1+1; 2+2+1; 2+1+1+1; 1+1+1+1+1.

(d) **TRUE.** $\mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/18\mathbb{Z}$ has elementary divisors $4, 2$ (at $p = 2$) and $3, 9$ (at $p = 3$); $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/36\mathbb{Z}$ has $2, 4$ and $3, 9$. The lists agree, so the groups are isomorphic. (Both have invariant factors $6 \mid 36$.)

(e) **TRUE.** $|S_4| = 24 = 2^3 \cdot 3$, so $n_2 \mid 3$ and n_2 is odd, giving $n_2 \in \{1, 3\}$. A Sylow 2-subgroup has order 8; the three copies of D_8 stabilising the three ways of pairing up the four points are distinct, so $n_2 = 3$.

(f) **TRUE.** An automorphism of a cyclic group is determined by the image of a generator, and the image may be any generator. Hence $\mathrm{Aut}(\mathbb{Z}/18\mathbb{Z}) \cong (\mathbb{Z}/18\mathbb{Z})^\times$ and $|\mathrm{Aut}(\mathbb{Z}/18\mathbb{Z})| = \varphi(18) = 18(1 - 1/2)(1 - 1/3) = 6$.

Grading note. Most of the credit here is in the *reason*, and the reason must be specific: a named counterexample, the actual permutation, or the actual count. A circled answer with no reason earns two of the five points; a wrong circle with a reason that is right about the underlying fact earns the three reason-points. Give full credit for a correct counterexample even if it is not the one printed above. The parenthetical remarks in each answer flag the error to expect on that item.

Problem 2. (25 points) **A group of order 195.**

(a) $|G| = 3 \cdot 5 \cdot 13$.

- $n_{13} \mid 15$ and $n_{13} \equiv 1 \pmod{13}$: the divisors $1, 3, 5, 15$ reduce to $1, 3, 5, 2$ mod 13, so $n_{13} = 1$.
- $n_5 \mid 39$ and $n_5 \equiv 1 \pmod 5$: the divisors $1, 3, 13, 39$ reduce to $1, 3, 3, 4$ mod 5, so $n_5 = 1$.
- $n_3 \mid 65$ and $n_3 \equiv 1 \pmod 3$: the divisors $1, 5, 13, 65$ reduce to $1, 2, 1, 2$ mod 3, so $n_3 \in \{1, 13\}$.

(b) Let $P \in \mathrm{Syl}_5(G)$ and $R \in \mathrm{Syl}_{13}(G)$; by (a) both are normal in G. Then PR is a subgroup (a product of subgroups is a subgroup as soon as one of them is normal), and since $|P \cap R|$ divides $\gcd(5, 13) = 1$,

$$|PR| = \frac{|P|\,|R|}{|P \cap R|} = 5 \cdot 13 = 65.$$

It is normal: for any $g \in G$, $g(PR)g^{-1} = (gPg^{-1})(gRg^{-1}) = PR$. So $H := PR \trianglelefteq G$ has order 65.

(c) Consider

$$1 \ \trianglelefteq\ R \ \trianglelefteq\ H \ \trianglelefteq\ G.$$

Each containment is legitimate: $R \trianglelefteq G$ so certainly $R \trianglelefteq H$, and $H \trianglelefteq G$ by (b). The successive quotients are

$$R \cong \mathbb{Z}/13\mathbb{Z}, \qquad H/R \text{ of order } 5 \cong \mathbb{Z}/5\mathbb{Z}, \qquad G/H \text{ of order } 3 \cong \mathbb{Z}/3\mathbb{Z},$$

all abelian (indeed cyclic of prime order). Hence G is solvable.

Grading note. In (a) award 2 points per prime, and only for a stated congruence — the answer $n_{13} = 1$ with no reduction of 15 mod 13 is worth 1. In (b) the two things to check are that PR is a *subgroup* (normality of one factor) and that $|PR| = 65$ (trivial intersection); a solution that asserts $|PR| = 65$ without the intersection argument gets 5 of 8. In (c) it is enough to exhibit a chain with abelian quotients; students need not verify simplicity of the factors, since solvability does not require it.

Problem 3. (25 points) **Invariant factors and elementary divisors.**

(a) $20 = 2^2 \cdot 5$, $30 = 2 \cdot 3 \cdot 5$, $50 = 2 \cdot 5^2$, so

$$|A| = 20 \cdot 30 \cdot 50 = 30000 = 2^4 \cdot 3 \cdot 5^4.$$

Splitting each cyclic factor into its prime-power parts gives the elementary divisors

$$\underbrace{4,\ 2,\ 2,}_{p=2} \qquad \underbrace{3}_{p=3}, \qquad \underbrace{25,\ 5,\ 5}_{p=5}.$$

The longest of these lists has three entries, so there are three invariant factors. Align the lists from the largest, padding with 1s, and multiply down the columns:

	col 1	col 2	col 3
$p = 2$	4	2	2
$p = 3$	3	1	1
$p = 5$	25	5	5
	300	10	10

so the invariant factors are $10 \mid 10 \mid 300$, i.e. $A \cong \mathbb{Z}_{10} \times \mathbb{Z}_{10} \times \mathbb{Z}_{300}$. (Check: $10 \cdot 10 \cdot 300 = 30000$.)

(b) The number of abelian groups of order $\prod p_i^{a_i}$ is $\prod p(a_i)$, where $p(a)$ is the number of partitions of a. Here $30000 = 2^4 \cdot 3^1 \cdot 5^4$, so the count is $p(4) \cdot p(1) \cdot p(4) = 5 \cdot 1 \cdot 5 = \boxed{25}$.

(c) $392 = 2^3 \cdot 7^2$. The partitions of 3 give 2-parts (8), $(4, 2)$, $(2, 2, 2)$; the partitions of 2 give 7-parts (49), $(7, 7)$. That is $3 \cdot 2 = 6$ groups:

elementary divisors	invariant factors
$8;\ 49$	392
$8;\ 7, 7$	$7 \mid 56$
$4, 2;\ 49$	$2 \mid 196$
$4, 2;\ 7, 7$	$14 \mid 28$
$2, 2, 2;\ 49$	$2 \mid 2 \mid 98$
$2, 2, 2;\ 7, 7$	$2 \mid 14 \mid 14$

Grading note. Part (a): 2 points for the factorisation of the order, 3 for the elementary divisors, 3 for the invariant factors. The single most common error is to align the prime lists from the *left* without padding, which produces $4 \cdot 3 \cdot 25 = 300$ in the wrong column or an incorrect number of factors; insist on the "align from the largest" rule. In (c) each line is worth 1.33 points; a student who lists elementary divisors only and does not convert should get 4 of the 8.

Problem 4. (20 points) **A direct product.**

Throughout, $Z(G_1 \times G_2) = Z(G_1) \times Z(G_2)$, and a Sylow p-subgroup of $G_1 \times G_2$ is exactly a product $P_1 \times P_2$ of Sylow p-subgroups of the factors — so $n_p(G_1 \times G_2) = n_p(G_1)\, n_p(G_2)$.

(a) $|G| = 12 \cdot 15 = 180 = 2^2 \cdot 3^2 \cdot 5$. Since $Z(A_4) = 1$ (no nonidentity element of A_4 commutes with everything: a 3-cycle is moved by conjugation, and a double transposition is moved by a 3-cycle) and $\mathbb{Z}/15\mathbb{Z}$ is abelian,

$$Z(G) = Z(A_4) \times Z(\mathbb{Z}/15\mathbb{Z}) = 1 \times \mathbb{Z}/15\mathbb{Z} \cong \mathbb{Z}/15\mathbb{Z}.$$

(b) A Sylow 3-subgroup of G has order 9 and is $P_1 \times P_2$ with $P_1 \in \mathrm{Syl}_3(A_4)$ (order 3) and $P_2 \in \mathrm{Syl}_3(\mathbb{Z}/15\mathbb{Z})$ (order 3). Now $n_3(A_4) = 4$ — the four subgroups generated by 3-cycles — and $n_3(\mathbb{Z}/15\mathbb{Z}) = 1$ since $\mathbb{Z}/15\mathbb{Z}$ is abelian. Hence $n_3(G) = 4 \cdot 1 = \boxed{4}$.

(c) The order of $(a, b) \in A_4 \times \mathbb{Z}/15\mathbb{Z}$ is $\mathrm{lcm}(|a|, |b|)$. The element orders in A_4 are $1, 2, 3$; those in $\mathbb{Z}/15\mathbb{Z}$ are $1, 3, 5, 15$. The largest lcm available is $\mathrm{lcm}(2, 15) = 30$, attained by e.g. $((1\,2)(3\,4), \bar{1})$. So the answer is $\boxed{30}$. (Note 180 is not attained: G is not cyclic.)

(d) A Sylow 2-subgroup of G has order 4 and equals $V_4 \times 1$, since V_4 is the unique Sylow 2-subgroup of A_4 ($V_4 \trianglelefteq A_4$) and $\mathbb{Z}/15\mathbb{Z}$ has odd order. Hence $n_2(G) = 1 \cdot 1 = \boxed{1}$.

Grading note. The two facts being tested are D&F §5.1 #1 and #4, and students who know them finish in four minutes. The expected errors: in (a), writing $Z(G) = 1$ because A_4 has trivial centre — the centre of a product is a product, not an intersection; in (c), answering 180 (assuming G is cyclic because gcds look coprime) or 15 (forgetting to take the lcm with an element of order 2). Award 2 of 4 in (b) for the correct value of $n_3(A_4)$ alone.