MATH 3551 A — Abstract Algebra I

Suggested Problems — Version 2

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What changed from Version 1 (September 28, 2026). Version 2 adds selected problems from Chapter 0 to make the foundational material easier to practice. Material from §§0.1–0.3 is included on Midterm 1. The exam groupings have also been updated to enforce the one-week coverage freeze: §1.7, §2.2 and §§3.1–3.2 move from Midterm 1 to Midterm 2, while §4.4 moves from Midterm 2 to Midterm 3. Sections 1.4 and 1.5 are required independent reading. No Version 1 problems were deleted; Version 1 remains available on the course webpage for direct comparison.

How to use this list

These problems are not collected and they are not graded. There is no graded homework in this course. Nothing here counts toward your grade directly, and everything here counts toward it indirectly, because the exams are drawn from this material.

Each section has two parts.

Warm-up problems come first. They are computations in a *specific* group — D_8, S_4, $\mathbb{Z}/12\mathbb{Z}$, Q_8, $(\mathbb{Z}/20\mathbb{Z})^\times$ — and most of them are a few lines or a table. They are written out in full below, so you do not need the book to start. **Do these first.** Nobody develops any feel for this subject by proving theorems about arbitrary groups before they can compute in three or four real ones.

Dummit and Foote lists five to nine exercises per section from the textbook. These are where the proof-writing lives. They are chosen to be the accessible ones: you will notice that the numbers stay small, and that whole stretches of each exercise set are not listed. That is deliberate. D&F is used for both undergraduate and graduate courses, and the back half of most of its exercise sets is aimed at the graduate one. If a problem you find in the book looks unreasonable, it may well be, and it is probably not on this list.

The exam problems will be similar but not identical. I am not going to reprint these problems on a midterm. I will ask something of the same kind: the same definition, the same argument shape, a different group or a different number. The goal is to handle a problem of this kind, cold, in fifty minutes, without the book.

How much of this to do. The rule of thumb is about three hours of work outside of class for every hour spent in it. This course meets three hours a week, so plan on roughly nine hours a week — reading the section, then the warm-ups, then the Dummit and Foote problems. This list is sized for that. If two sections are covered in a week, that is two sets of warm-ups and ten to eighteen exercises, which is what nine hours buys at a reasonable pace.

Work in groups, and attempt before you look anything up. Explaining a proof out loud to somebody who is not convinced is the fastest way to find out that you do not actually believe it yourself. And a problem you have struggled with for fifteen minutes and then seen solved teaches you something; a problem whose solution you read first teaches you almost nothing, and it feels exactly the same at the time. Nothing here is submitted, so collaborate freely.

What is on which exam

- **Midterm 1**, Monday October 12 — §§0.1–0.3, §§1.1–1.6, §2.1 and §§2.3–2.5.
- **Midterm 2**, Friday November 6 — §1.7, §2.2 and §§3.1–4.3.
- **Midterm 3**, Monday November 30 — §§4.4–5.2.
- **Final**, Thursday December 17 — cumulative, and the only exam that reaches §5.4, §5.5 and §6.1.

§5.3 (the table of groups of small order) is not on the syllabus. It is four pages, it is worth reading, and nothing will be asked about it.

Part I. Early course material (§§0.1–3.2)

The sections in this part remain in textbook order. The exam-status boxes and the list above show which sections belong to Midterm 1 and which have moved to Midterm 2.

§0.1 Basics

Sets, functions and equivalence relations — the language used to state the group-theory definitions precisely.

Warm-up. Write down one injective non-surjective function and one surjective non-injective function between small finite sets. Then give an equivalence relation on a six-element set with classes of sizes 1, 2 and 3.

Dummit and Foote. 4, 6

Exam status. Included on Midterm 1. These refresher problems were added in Version 2 to make the foundational material easier to practice.

§0.2 Properties of the Integers

Divisibility, greatest common divisors, the Euclidean algorithm and Bézout identities.

Warm-up. Use the Euclidean algorithm to compute $\gcd(252, 198)$, then write that gcd as an integer linear combination of 252 and 198.

Dummit and Foote. 1, 2, 3, 4, 5, 10, 11

Exam status. Included on Midterm 1.

§0.3 $\mathbb{Z}/n\mathbb{Z}$: The Integers Modulo n

Congruences, arithmetic modulo n, units and multiplicative inverses.

Warm-up. List the units in $\mathbb{Z}/12\mathbb{Z}$ and find the inverse of each. Then solve $5x \equiv 7 \pmod{12}$ and decide whether $6x \equiv 5 \pmod{12}$ has a solution.

Dummit and Foote. 3, 4, 5, 6, 7, 8, 9

Exam status. Included on Midterm 1.

§1.1 Basic Axioms and Examples

The group axioms, orders of elements, and $\mathbb{Z}/n\mathbb{Z}$ and $(\mathbb{Z}/n\mathbb{Z})^\times$ — the examples every later chapter is built out of.

Warm-up. (a) Write out the full multiplication table of $(\mathbb{Z}/8\mathbb{Z})^\times = \{1, 3, 5, 7\}$. Find the order of each element. Is there an element of order 4? (b) Do the same for $(\mathbb{Z}/10\mathbb{Z})^\times = \{1, 3, 7, 9\}$. Find an element of order 4. (c) In $\mathbb{Z}/12\mathbb{Z}$ (under addition), find the order of every element. Which elements have order 12? (d) Decide, with a one-line reason, whether each is a group: the odd integers under addition; $\{1, -1, i, -i\}$ under multiplication; the positive rationals under multiplication; the 2×2 integer matrices with nonzero determinant, under multiplication. (e) Let $S = \mathbb{R} \setminus \{-1\}$ and define $a * b = a + b + ab$. Check the axioms one at a time and show $(S, *)$ is an abelian group. (What is the identity? What is a^{-1}?)

Dummit and Foote. 1, 5, 11, 20, 22, 25, 26, 32

Note. Exercise 26 is the subgroup criterion. The text does not prove it and we use it constantly from §2.1 onward, so this exercise *is* the proof — do it. Exercise 32 (if $|x| = n$ then $1, x, \ldots, x^{n-1}$ are distinct) is four lines and gets quoted all term.

§1.2 Dihedral Groups

D_{2n} is the standing example of a nonabelian group. Learn to compute with r, s and the relation $rs = sr^{-1}$.

Warm-up. (a) List all eight elements of D_8 in the form r^i or sr^i, and find the order of each. (b) Compute, in D_8: $rs \cdot sr^2$, $(sr)^{-1}$, $(sr^3)^2$, r^2sr^{-1}. (c) Show by direct computation that $sr^i = r^{-i}s$ in D_{2n} for every i. (Induct on i using $sr = r^{-1}s$.) (d) How many elements of order 2 does D_8 have? How many does D_{10} have? Explain the difference in one sentence. (e) Draw a square, label its vertices $1, 2, 3, 4$, and write r and s as permutations in cycle notation. Check your answer to (b) against these permutations.

Dummit and Foote. 1, 2, 3, 4, 5, 6

Note. Exercise 3 (every element that is not a power of r has order 2) gets used repeatedly. Exercises 4 and 5 compute the center $Z(D_{2n})$, which §2.2 #7 depends on.

§1.3 Symmetric Groups

Cycle notation, and the fact that the order of a permutation is the lcm of its cycle lengths — the computational engine of the whole course.

Warm-up. (a) Let $\sigma = (1\,3\,5)(2\,4)$ and $\tau = (1\,2\,4\,5)$ in S_5. Compute $\sigma\tau$, $\tau\sigma$, σ^{-1}, σ^2 and τ^3, each as a product of disjoint cycles. (b) Find $|\sigma|$, $|\tau|$ and $|\sigma\tau|$. Is $|\sigma\tau|$ what you expected? (c) Write $(1\,2\,3\,4\,5)$ as a product of transpositions. Do it a second way with a different number of transpositions, if you can. (d) What orders occur among the elements of S_5? Of S_6? (List the cycle types and take lcms.) (e) How many 3-cycles are there in S_5? How many elements of order 2?

Dummit and Foote. 1, 2, 3, 4, 5, 14, 15, 18

Note. Exercise 15 (the order of a permutation is the lcm of its cycle lengths) is the source of a fact used on every exam after this one. Prove it once, properly, and you can quote it forever.

§1.4 Matrix Groups

Exam status. Included on Midterm 1 as required independent reading.

$GL_n(F)$ as a source of finite nonabelian groups, and a first look at the field $\mathbb{F}_p$.

Warm-up. (a) Write down two matrices in $GL_2(\mathbb{R})$ that do not commute, and verify it. (b) List all six invertible 2×2 matrices over $\mathbb{F}_2$. (There are exactly six.) Find the order of each. What familiar group is this? (c) Count $GL_2(\mathbb{F}_3)$ by counting ordered bases: how many choices for the first column, and then how many for the second? (d) Show that the set of 2×2 real matrices $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ is a group under multiplication, and find a formula for the inverse.

Dummit and Foote. 1, 2, 3, 4, 5, 7

Note. Exercise 7 (the order of $GL_2(\mathbb{F}_p)$) is the counting argument worth doing carefully. Do not just quote the formula in the text — warm-up (c) is the $p = 3$ rehearsal for it.

§1.5 The Quaternion Group

Exam status. Included on Midterm 1 as required independent reading.

Q_8: the other nonabelian group of order 8, and the standard counterexample to almost everything you might guess.

Warm-up. (a) Write out the full multiplication table of $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. (b) Find the order of every element of Q_8. How many elements of order 2 are there? (c) Compare with your answer for D_8 in §1.2 warm-up (d). Both groups have order 8 and both are nonabelian. Give a one-sentence reason why they cannot be isomorphic. (d) List every subgroup of Q_8. (There are six.) Notice that every one of them is normal, which is unusual and which we will come back to.

Dummit and Foote. 1, 2, 3

Note. Dummit and Foote give only three exercises here and doing all three takes twenty minutes. D_8 versus Q_8 is a recurring exam theme, so warm-up (c) is the one to be able to produce cold.

§1.6 Homomorphisms and Isomorphisms

What it means for two groups to be the same, and — much more often needed — the standard ways to prove that two groups are not.

Warm-up. (a) For each map, say whether it is a homomorphism, and if so whether it is an isomorphism: $\varphi : \mathbb{Z} \to \mathbb{Z}$, $\varphi(n) = 3n$, additive; $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(x) = x^3$, additive; $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(x) = x^3$, multiplicative on $\mathbb{R}^\times$; $\varphi : \mathbb{Z} \to \mathbb{Z}$, $\varphi(n) = n + 1$, additive. (b) Find all homomorphisms $\mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$. (Where can the generator $\bar{1}$ go?) (c) Find all homomorphisms $\mathbb{Z} \to \mathbb{Z}/12\mathbb{Z}$. (d) Show $(\mathbb{Z}/5\mathbb{Z})^\times \cong (\mathbb{Z}/10\mathbb{Z})^\times$ by writing down an explicit bijection and checking it. Then show $(\mathbb{Z}/12\mathbb{Z})^\times$ is isomorphic to neither. (e) Which of $\mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, D_8, Q_8, $\mathbb{Z}/2\mathbb{Z}^3$ are isomorphic to each other? Use orders of elements, and abelian-versus-not, and nothing else.

Dummit and Foote. 1, 2, 4, 6, 7, 9, 14, 20

Note. Exercise 2 (an isomorphism preserves orders of elements) is the workhorse behind every non-isomorphism proof in the course, and 4, 6, 7 and 9 are all applications of it — you should be able to produce any of them in two lines. Exercise 20 (that $\mathrm{Aut}(G)$ is a group) is the definition §4.4 is built on.

§1.7 Group Actions

Exam status. Not on Midterm 1. This section moves to Midterm 2 because it is introduced during the final week before Midterm 1.

The action axioms and the permutation representation — the single most important idea in Chapters 4 and 5, introduced here in its easiest form.

Warm-up. (a) Let D_8 act on the four vertices of the square. Write down the orbit and the stabilizer of vertex 1. Check that $|D_8| = |\text{orbit}| \cdot |\text{stabilizer}|$. (b) Now let D_8 act on the two *diagonals* of the square. Write the action as a homomorphism $D_8 \to S_2$ and find its kernel. (c) Let $G = \mathbb{Z}/6\mathbb{Z}$ act on itself by addition. What are the orbits? What are the stabilizers? (d) Let S_4 act on $\{1, 2, 3, 4\}$ in the obvious way. Find the stabilizer of the point 4 and say which familiar group it is. (e) Show that the additive group $\mathbb{R}$ acts on the plane by $r \cdot (x, y) = (x + ry,\ y)$. Describe the orbits geometrically.

Dummit and Foote. 1, 2, 3, 4, 5, 6, 12

Note. Exercise 4 defines the stabilizer and the kernel of an action, which is exactly §2.2's subject — do it before §2.2. Warm-up (e) was problem 3 on Dummit's Fall 2014 Exam I, verbatim.

§2.1 Subgroups: Definition and Examples

The subgroup criterion, and the habit of checking it quickly and correctly.

Warm-up. (a) List every subgroup of $\mathbb{Z}/12\mathbb{Z}$. List every subgroup of $\mathbb{Z}/7\mathbb{Z}$. (b) List every subgroup of S_3. Which are normal? (You will have six elements and six subgroups.) (c) Decide, with a reason: is $\{\sigma \in S_4 : \sigma(1) = 1\}$ a subgroup of S_4? Is $\{\sigma \in S_4 : \sigma(1) = 3\}$? (d) Is $\mathbb{Q}(i) = \{a + bi : a, b \in \mathbb{Q}\}$ a subgroup of $(\mathbb{C}, +)$? Is the set of 2×2 integer matrices with determinant 1 a subgroup of $GL_2(\mathbb{R})$? (e) Give an example of a group G and subgroups H, K with $H \cup K$ *not* a subgroup.

Dummit and Foote. 1, 2, 3, 4, 5, 6, 8, 10

Note. Exercise 8 ($H \cup K$ is a subgroup if and only if one contains the other) is a perennial true/false item; warm-up (e) is half of it. Exercise 4 is the example showing why "closed under the operation" is not enough in the infinite case — worth ten minutes even though it looks like a triviality.

§2.2 Centralizers and Normalizers, Stabilizers and Kernels

Exam status. Not on Midterm 1. This section moves to Midterm 2.

The four subgroups attached to a subset. Telling them apart, and computing them, is most of the work in this section.

Warm-up. (a) In D_8, compute $C_{D_8}(r)$, $C_{D_8}(s)$ and $Z(D_8)$. (b) In S_4, compute the centralizer of $(1\,2)$ and the centralizer of $(1\,2\,3)$. Check the sizes against $|S_4| = 24$. (c) Compute $Z(S_3)$, $Z(S_4)$ and $Z(Q_8)$. (d) Let $A = \{1, r^2\} \subseteq D_8$. Compute $C_{D_8}(A)$ and $N_{D_8}(A)$, and say which containment between them holds. (e) Compute $Z(GL_2(\mathbb{R}))$. (Guess the answer first, then prove it by testing against the two elementary matrices.)

Dummit and Foote. 1, 2, 3, 4, 5, 6, 7, 10

Note. All fourteen exercises in this section are reasonable; these eight are the ones to do first. Exercise 5 (with $G = D_8$ and $A = \{1, s, r^2, sr^2\}$) is the standard exam computation here. Exercise 6(b), $H \leq C_G(H)$ if and only if H is abelian, gets quoted later.

§2.3 Cyclic Groups and Cyclic Subgroups

The complete classification of cyclic groups and of all of their subgroups — the one family of groups about which you should be able to answer any question asked.

Warm-up. (a) Find every generator of $\mathbb{Z}/12\mathbb{Z}$. Of $\mathbb{Z}/13\mathbb{Z}$. Of $\mathbb{Z}/20\mathbb{Z}$. (b) List every subgroup of $\mathbb{Z}/24\mathbb{Z}$, give a generator of each, and draw the containments. (c) Let $G = \langle x \rangle$ be cyclic of order 30. Find $|x^{12}|$, $|x^{18}|$ and $|x^{25}|$. How many subgroups does G have? (d) Is $(\mathbb{Z}/9\mathbb{Z})^\times$ cyclic? Is $(\mathbb{Z}/8\mathbb{Z})^\times$? Give a generator or a reason. (e) Prove or disprove: if $p \neq q$ are primes, $\mathbb{Z}/pq\mathbb{Z}$ has exactly $(p-1)(q-1)$ generators. (Counting the *non*-generators is easier.)

Dummit and Foote. 1, 2, 3, 4, 5, 8, 10, 16

Note. Exercise 16 ($|xy|$ divides $\operatorname{lcm}(|x|, |y|)$ when x and y commute) is used constantly in Chapter 5, and the hypothesis that they commute is not decoration — find a counterexample without it in S_3. Dummit's Fall 2014 Exam I problem 5 is warm-up (c) for a cyclic group of order 270.

§2.4 Subgroups Generated by Subsets of a Group

$\langle A\rangle$ as the smallest subgroup containing A, and recognizing a familiar group from a generating set.

Warm-up. (a) In S_4, compute $\langle (1\,2),\, (3\,4)\rangle$ and $\langle (1\,2),\, (1\,2\,3\,4)\rangle$. How big is each? (b) In $\mathbb{Z}/12\mathbb{Z}$, compute $\langle 4,\, 6\rangle$ and $\langle 8,\, 9\rangle$. (c) In D_8, show that $\langle s,\, sr\rangle = D_8$. (d) Show that $\mathbb{Z} = \langle 2,\, 3\rangle$ but $\mathbb{Z} \neq \langle 2,\, 4\rangle$.

Dummit and Foote. 1, 2, 3, 6

Note. Most of the rest of this exercise set is about infinite generation and is well beyond what we need. Four is enough here.

§2.5 The Lattice of Subgroups of a Group

Drawing the subgroup lattice, and reading facts about a group off the picture.

Warm-up. (a) Draw the subgroup lattice of $\mathbb{Z}/12\mathbb{Z}$, of $\mathbb{Z}/16\mathbb{Z}$ and of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. (b) Draw the subgroup lattice of D_8. (There are ten subgroups.) (c) Draw the subgroup lattice of Q_8 and compare it with the one for D_8. They are different pictures — say in one sentence how. (d) Draw the subgroup lattice of S_3.

Dummit and Foote. 1, 2, 3, 4, 5, 6

Note. These are drawings, not proofs, and they are worth the hour. Almost every structural fact about the small groups in this course is visible in one of these six pictures.

§3.1 Quotient Groups and Homomorphisms: Definitions and Examples

Exam status. Not on Midterm 1. This section moves to Midterm 2.

Kernel, image, normality, and the quotient group G/N — with G/N met first as a set of cosets you can actually list.

Warm-up. (a) Let $G = \mathbb{Z}/12\mathbb{Z}$ and $N = \langle 4\rangle = \{0, 4, 8\}$. List the cosets of N and write out the addition table of G/N. Which familiar group is it? (b) Let $G = S_3$ and $N = A_3 = \{1, (1\,2\,3), (1\,3\,2)\}$. List the cosets and give the multiplication table of G/N. (c) Let $G = D_8$ and $N = \langle r^2\rangle = \{1, r^2\}$. List the four cosets and identify G/N. (d) For the homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$, $\varphi(n) = \bar{n}$, find $\ker\varphi$ and $\operatorname{im}\varphi$, and check the first isomorphism theorem by hand. (e) Compute the kernel of $\det : GL_2(\mathbb{R}) \to \mathbb{R}^\times$ and name it.

Dummit and Foote. 1, 2, 3, 4, 6, 7, 14, 36, 37

Note. This is the largest and gentlest exercise set in Chapter 3; more of it is usable than in any other section. Exercise 3 (a quotient of an abelian group is abelian) plus an example of a nonabelian group with abelian quotient is the pair to have ready.

§3.2 More on Cosets and Lagrange's Theorem

Exam status. Not on Midterm 1. This section moves to Midterm 2.

Lagrange's theorem and the arithmetic it makes available — the first theorem in the course that lets you rule things out by counting.

Warm-up. (a) List the left cosets and the right cosets of $\langle 8\rangle$ in $\mathbb{Z}/24\mathbb{Z}$; of $3\mathbb{Z}$ in $\mathbb{Z}$; of A_4 in S_4; of $\langle (1\,2)\rangle$ in S_3. In which of these do the left and right cosets differ? (b) What are the possible orders of subgroups of a group of order 60? Of order 100? (c) Why must a finite group containing an element of order 5 and an element of order 7 have at least 35 elements? (d) Show that a group of prime order is cyclic, and that any nonidentity element generates it. (e) In S_4, find a subgroup of order 8 and a subgroup of order 3. Is there a subgroup of order 6? (Try to build one; Lagrange does not forbid it.)

Dummit and Foote. 1, 2, 4, 5, 6, 8, 11, 16

Note. Warm-up (e) is a preview: A_4 has order 12 and no subgroup of order 6, so the converse of Lagrange's theorem is false. We will prove that in §3.5. Exercise 8 ($H \cap K = 1$ when $|H|$ and $|K|$ are coprime) is three lines and gets used all term.

Part II. Middle course material (§§3.3–4.4)

Sections 3.3 through 4.3 are on Midterm 2. Section 4.4 moves to Midterm 3.

§3.3 The Isomorphism Theorems

Four theorems that say a diagram you already believe is actually an isomorphism. The work is in setting up the map, not in the proof.

Warm-up. (a) In $\mathbb{Z}/24\mathbb{Z}$ let $H = \langle 4\rangle$ and $N = \langle 6\rangle$. List HN and $H \cap N$. Then list the cosets of N in HN, and the cosets of $H \cap N$ in H, and match them up. That is the second isomorphism theorem, done by hand. (b) In $\mathbb{Z}/12\mathbb{Z}$ with $N = \langle 6\rangle$: list the subgroups of $\mathbb{Z}/12\mathbb{Z}$ containing N, list the subgroups of $\mathbb{Z}/12\mathbb{Z}/N$, and match them up. That is the fourth (lattice) isomorphism theorem. (c) Use the first isomorphism theorem to show $\mathbb{R}^\times/\{\pm 1\} \cong \mathbb{R}_{>0}$. (d) Use the first isomorphism theorem to show $S_4/V_4 \cong S_3$, where $V_4 = \{1, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$. (Let S_4 act on the three ways of splitting $\{1, 2, 3, 4\}$ into two pairs.)

Dummit and Foote. 1, 2, 3, 7

Note. Warm-up (a) is the whole content of the second isomorphism theorem in a group small enough to write down completely. Do it before you read the proof; the proof will then be obvious.

§3.4 Composition Series and the Hölder Program

Breaking a finite group into simple pieces, and the statement — not the proof — of Jordan–Hölder.

Warm-up. (a) Write down a composition series for $\mathbb{Z}/12\mathbb{Z}$, and another one with the factors in a different order. Check that the multiset of factors is the same. (b) Write down a composition series for S_4. (Use $S_4 \rhd A_4 \rhd V_4 \rhd \mathbb{Z}/2\mathbb{Z} \rhd 1$ and identify each quotient.) (c) Write down a composition series for D_8 and identify the factors. (d) Show directly that $\mathbb{Z}/p\mathbb{Z}$ is simple for p prime, and that $\mathbb{Z}/4\mathbb{Z}$ is not.

Dummit and Foote. 1, 2, 3, 4, 5

Note. Jordan–Hölder is quoted, not proved, in this course. What you are responsible for is producing a composition series for a specific small group and naming its factors, which is exactly the warm-up.

§3.5 Transpositions and the Alternating Group

The sign of a permutation, and A_n — including the standard example showing the converse of Lagrange's theorem is false.

Warm-up. (a) Write each of $(1\,2\,3)$, $(1\,2\,3\,4)$, $(1\,2)(3\,4)$, $(1\,2\,3\,4\,5)$ as a product of transpositions, and give the sign of each. (b) List the twelve elements of A_4 by cycle type. (c) Find every subgroup of A_4 and its order. Conclude that A_4 has no subgroup of order 6, so the converse of Lagrange's theorem is false. (d) What cycle types occur in A_5? What orders of elements occur? (e) Show that σ^2 is even for every $\sigma \in S_n$.

Dummit and Foote. 1, 2, 3, 4, 5, 6, 7

Note. Warm-up (c) is the single most quoted example in the second half of the course. Exercise 7 (the rigid motions of a regular tetrahedron form A_4) is the geometric version of the same group and is worth doing with a physical tetrahedron if you can find one.

§4.1 Group Actions and Permutation Representations

Orbits, stabilizers, and the orbit–stabilizer theorem — Lagrange's theorem wearing a different hat.

Warm-up. (a) Let S_4 act on $\{1, 2, 3, 4\}$. Compute the orbit and the stabilizer of the point 2 and check orbit–stabilizer. (b) Let D_{12} (the symmetries of a regular hexagon) act on the six vertices. Compute the orbit and stabilizer of one vertex. Now let it act on the three long diagonals: what are the orbits and stabilizers? (c) Let $\langle(1\,2\,3\,4)\rangle \leq S_4$ act on $\{1, 2, 3, 4\}$. What are the orbits? (d) Let G act on itself by conjugation. Show the orbit of z is $\{z\}$ exactly when $z \in Z(G)$. Then compute all the conjugation orbits in D_8.

Dummit and Foote. 1, 2, 3, 4, 7, 8, 10

Note. Warm-up (d) is the bridge to §4.3: the class equation is nothing but orbit–stabilizer applied to the conjugation action, with the size-one orbits collected on one side.

§4.2 Cayley's Theorem

Every group is a group of permutations. The proof is short; the value is in the examples.

Warm-up. (a) Write out the left regular representation of $\mathbb{Z}/4\mathbb{Z}$ as a subgroup of S_4: give each of the four elements as a permutation in cycle notation. (b) Do the same for the Klein four-group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, again inside S_4. Compare with (a) — both are order-4 subgroups of S_4, but different ones. (c) Write out the left regular representation of S_3 inside S_6 for the two elements $(1\,2)$ and $(1\,2\,3)$. (d) Using (a) and (b), explain why every group of order n embeds in S_n but usually not in anything much smaller.

Dummit and Foote. 1, 2, 3, 4, 7, 8, 9, 10

Note. Exercises 1 through 4 are all short. Exercise 9 (a group of order $2n$ with n odd has a subgroup of index 2) is the one real theorem in the section and it is a nice application of the regular representation — try it after the others.

§4.3 The Class Equation

Conjugacy classes, and the counting argument that gives the first nontrivial structure theorem in the course: a p-group has a nontrivial center.

Warm-up. (a) Find all the conjugacy classes of S_3 and write down its class equation. (b) Do the same for S_4. (Conjugacy classes in S_n are cycle types — use that.) (c) Do the same for D_8 and for Q_8. Both have order 8; compare the two class equations. (You should get the same one twice — so the class equation does not determine the group.) (d) Write down the class equation of $\mathbb{Z}/9\mathbb{Z}$, and of any abelian group of order n. (e) How many elements are conjugate to $(1\,2\,3)$ in S_5? To $(1\,2)(3\,4)$ in S_5?

Dummit and Foote. 1, 2, 3, 4, 5, 6, 13

Note. Warm-up (b) — conjugacy in S_n is determined by cycle type — is the most-used fact in this section, and the counting in (e) shows up on exams. The class equations you write in (a)–(d) are exactly what makes the p-group theorem readable.

§4.4 Automorphisms

Exam status. Not on Midterm 2. This section moves to Midterm 3 because it is scheduled during the final week before Midterm 2.

$\mathrm{Aut}(G)$ *for the small groups we know. This section is short in this course.*

Warm-up. (a) Compute $\mathrm{Aut}(\mathbb{Z}/6\mathbb{Z})$ and $\mathrm{Aut}(\mathbb{Z}/8\mathbb{Z})$. (An automorphism is determined by where the generator goes, and it must go to a generator.) (b) Show $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$ in general, using the idea in (a). (c) Compute $\mathrm{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ and identify it. (It permutes the three nonidentity elements.) (d) Show that conjugation by a fixed $g \in G$ is an automorphism of G, and that it is the identity automorphism exactly when $g \in Z(G)$.

Dummit and Foote. 1, 2, 3, 4

Note. Most of this exercise set computes automorphism groups of larger and stranger groups and is well past what we need. Warm-ups (a)–(c) are the three computations to be able to do.

Part III. Later course material (§§4.5–5.2)

Together with §4.4 above, these sections make up the Midterm 3 coverage window.

§4.5 The Sylow Theorems

Existence, conjugacy and counting of Sylow p-subgroups, and the standard argument that a group of a given order cannot be simple. This is the technical high point of the course.

Warm-up. (a) For a group of order 18, 24, 54, 72 and 80: what is the order of a Sylow p-subgroup for each prime p dividing the order? (b) Find all the Sylow 2-subgroups and all the Sylow 3-subgroups of S_3, of D_{12}, and of A_4. Count them and check the count against $n_p \equiv 1 \pmod p$ and $n_p \mid m$. (c) Find all the Sylow 3-subgroups of S_4 and verify directly that two of them are conjugate. (d) Let $|G| = 15$. Show $n_3 = n_5 = 1$, and deduce that G is cyclic. (e) Let $|G| = 45$. Show G has a normal subgroup of order 9. (f) Let $|G| = 33$. Show G has exactly one subgroup of order 3.

Dummit and Foote. 1, 2, 3, 4, 5, 13, 19, 24, 26

Note. The pattern in warm-ups (d), (e), (f) — write down the possible n_p, rule out all but one, conclude normality — is the single most examinable technique in the course, and exercises 13, 19, 24 and 26 are the same pattern for orders 56, 6545, 231 and 105. Do all four.

This exercise set runs to 56 problems and the back half of it is a graduate course. Nothing past #26 is expected of you.

§4.6 The Simplicity of A_n

A_5 is simple, and A_n is simple for $n \geq 5$. Mostly a lecture section.

Warm-up. (a) Write down the conjugacy classes of A_5 and their sizes. (They are $1, 15, 20, 12, 12$.) (b) A normal subgroup is a union of conjugacy classes containing the identity, and its order divides 60. Check that no sum $1 +$ (some of $15, 20, 12, 12$) divides 60 except 1 and 60. That is a complete proof that A_5 is simple, and it is worth writing out. (c) Why does (b) show that a group of order 60 with more than one Sylow 5-subgroup and more than one Sylow 3-subgroup is a candidate to be A_5?

Dummit and Foote. 1, 2

Note. Dummit and Foote give only a handful of exercises here and they are all long. The simplicity of A_5 is what you are responsible for, and warm-up (b) is the whole argument. The general A_n case is quoted, not proved.

§5.1 Direct Products

Building new groups from old, and telling when $A \times B$ is something you already know.

Warm-up. (a) List all eight elements of $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and find the order of each. Is it cyclic? (b) Find the order of $(3, 4)$ in $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$; of $(6, 15, 4)$ in $\mathbb{Z}/30\mathbb{Z} \times \mathbb{Z}/45\mathbb{Z} \times \mathbb{Z}/24\mathbb{Z}$. (c) Show $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z}$ but $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \not\cong \mathbb{Z}/4\mathbb{Z}$. State the general rule you have just illustrated. (d) In $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, compute the cyclic subgroup generated by each element. How many subgroups of order 3 are there? (e) Is $A_4 \times \mathbb{Z}/2\mathbb{Z} \cong S_4$? Both have order 24. Justify your answer.

Dummit and Foote. 1, 2, 3, 4

Note. Warm-up (c) is the Chinese Remainder Theorem for groups and it is the fact you will use most often in §5.2. Exercise 1 (the center of a product is the product of the centers) is three lines and gets quoted.

§5.2 The Fundamental Theorem of Finitely Generated Abelian Groups

The complete classification of finite abelian groups. Once you can do the bookkeeping, every question about a finite abelian group is answerable.

Warm-up. (a) List all abelian groups of order 8 up to isomorphism. Of order 16. Of order 36. (b) List all abelian groups of order p^2q for distinct primes p, q. (c) Give the invariant factors and the elementary divisors of $\mathbb{Z}/12\mathbb{Z}\times\mathbb{Z}/18\mathbb{Z}$. (d) Which of these are isomorphic? $\mathbb{Z}/36\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/12\mathbb{Z}\times\mathbb{Z}/18\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/9\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}\times\mathbb{Z}/18\mathbb{Z}$. (All have order 216.) (e) Show that an abelian group of order 252 has an element of order 42.

Dummit and Foote. 1, 2, 3

Note. Exercises 1, 2 and 3 are the classification computations and they are exactly right for this course — do all of them. Warm-up (d) is Dummit's Fall 2014 Exam II problem 4(b) with smaller numbers, and something of that shape is very likely on Midterm 3.

Part IV. Final only (§5.4, §5.5, §6.1)

§5.4 Recognizing Direct Products

When is a group the direct product of two of its subgroups? The criterion, and how to check it.

Warm-up. (a) Show $\mathbb{Z}/6\mathbb{Z} = \langle 2\rangle \times \langle 3\rangle$ by checking the criterion: both normal, intersection trivial, product everything. (b) Show that D_{12} (order 12) is $\mathbb{Z}/2\mathbb{Z} \times S_3$ by exhibiting the two subgroups and checking the criterion. (c) Show that D_8 is *not* a direct product of two proper subgroups. (Where does the criterion fail?) (d) Compute the commutator $[r, s]$ in D_8, and then the commutator subgroup D_8'. (e) Compute S_3' and S_4'.

Dummit and Foote. 1, 2, 3, 4

Note. The commutator subgroup is defined in this section and used in §6.1. Warm-ups (d) and (e) are the two computations to have.

§5.5 Semidirect Products

The construction that produces the nonabelian groups of small order. We do the construction and the two standard examples; the classification theorems are not ours.

Warm-up. (a) Write D_8 as a semidirect product $\mathbb{Z}/4\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$: give the two subgroups and the homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/4\mathbb{Z})$. (b) Write S_3 as $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ in the same way. (c) Show that the only homomorphism $\mathbb{Z}/3\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/2\mathbb{Z})$ is trivial, and conclude that every group of order 6 with a normal Sylow 2-subgroup is cyclic. (d) Explain why Q_8 is *not* a semidirect product of two proper subgroups.

Dummit and Foote. 1, 2

Note. Semidirect products are where an undergraduate first course usually stops, and the exercise set here is aimed well past us. The four warm-ups are what is expected.

§6.1 p-groups, Nilpotent Groups, and Solvable Groups

Solvability and nilpotence, met through the examples we already have. Almost entirely a lecture section.

Warm-up. (a) Show S_3 is solvable by exhibiting the chain $1 \lhd A_3 \lhd S_3$ and naming the abelian quotients. (b) Show S_4 is solvable using $1 \lhd \mathbb{Z}/2\mathbb{Z} \lhd V_4 \lhd A_4 \lhd S_4$. (c) Show D_{2n} is solvable for every n, using $1 \lhd \langle r\rangle \lhd D_{2n}$. (d) Show A_5 is not solvable, in one line, from its simplicity. (e) Compute the lower central series of D_8 and conclude D_8 is nilpotent of class 2. Then compute the derived series of A_4.

Dummit and Foote. 1, 2

Note. No undergraduate course I know of assigns much from this exercise set, and we will not either. Warm-ups (a)–(e) are the whole of what is expected: produce a chain with abelian quotients for a specific small group, and know that A_5 has none.