Math 163 — Test 01

Monday September 17th 2012

Instructions Remember to show all your work so you can get partial credit. You shouldn't need a calculator on this test. Please leave answers in their exact form. Try not to overthink the problems too much.

1. (15 Points) Find the derivatives of the following functions
   (a) $f(x) = \ln(x)$.
   (b) $h(t) = e^{t+1}$.
   (c) $g(x) = 2^x$

a) $\frac{d}{dx} \ln(x) = 1/x$

b) $\frac{d}{dx}\left[e^{t+1}\right] = e^{t+1}$

c) $\frac{d}{dx}\left[2^x\right] = \frac{d}{dx}\left[e^{\ln(2)x}\right] = \ln(2)e^{\ln(2)x}$

2. (15 Points) Find the following integrals

(a) $\int_1^x \frac{1}{t} dt$

(b) $\int \frac{1}{1+4x^2} dx$

(c) $\int \tan(\theta) d\theta$

a)

$$\int_1^x \frac{1}{t} dt = \ln(x)$$

b)

$$\int \frac{1}{1+4x^2} dx = \int \frac{1}{1+(2x)^2} dx \ , \quad \boxed{\begin{array}{l} u = 2x \\ du = 2\cancel{x}dx \\ \frac{du}{2} = dx \end{array}}$$

$$= \int \frac{1}{1+u^2} \frac{du}{2} = \frac{1}{2}\left[\tan^{-1}(u) \right] + C$$

$$= \frac{1}{2} \tan^{-1}(2x) + C$$

c)

$$\int \frac{\sin(\theta)}{\cos(\theta)} d\theta = \int -\frac{du}{u} = -\ln|u| + c$$

$$= -\ln|\cos(\theta)| + c$$

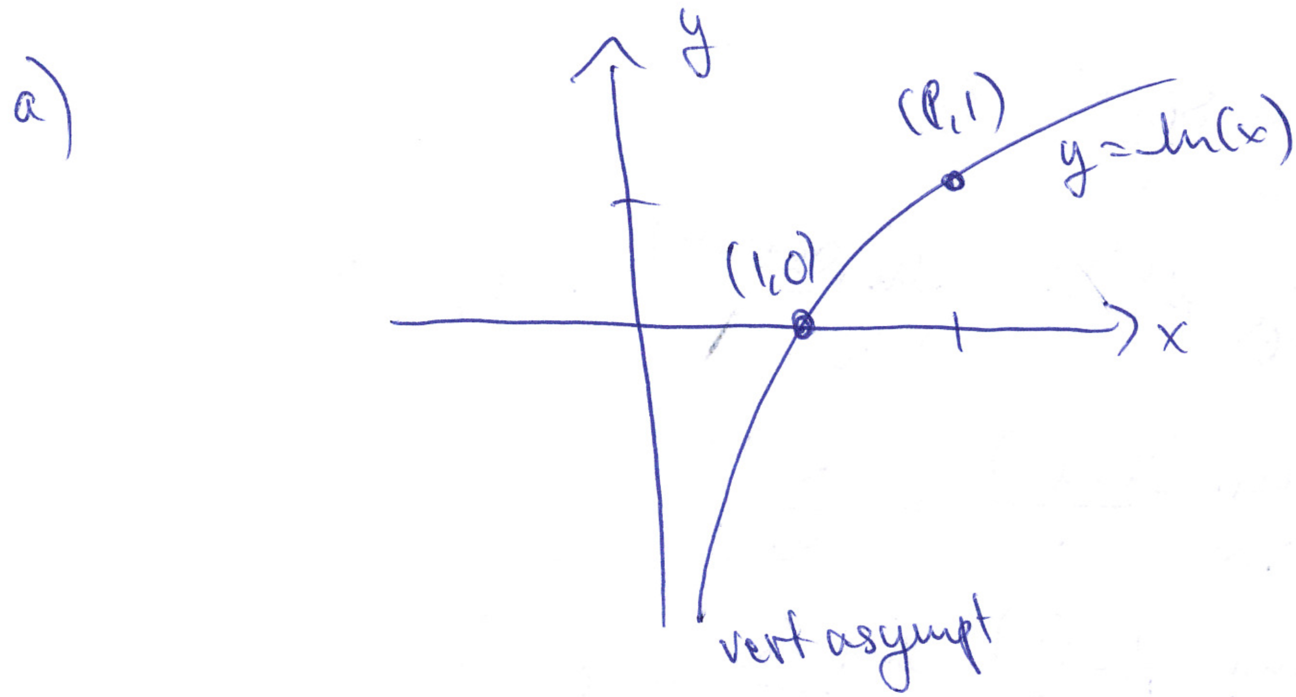
$u = \cos(\theta)$

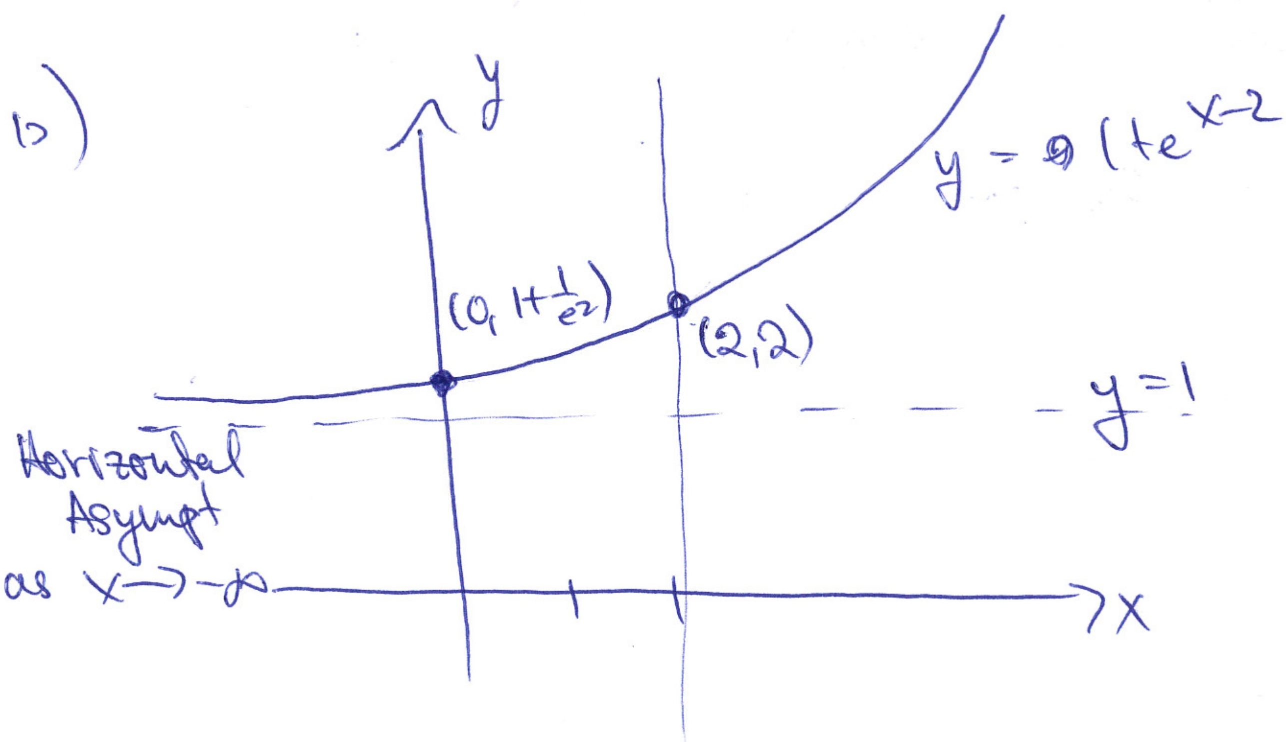
$du = -\sin(\theta) d\theta$

3. (10 Points) Graph the following functions. Make sure to label key features.

(a) $f(x) = \ln(x)$.

(b) $f(x) = 1 + e^{x-2}$.





4. (10 Points) Find the following limits using L'hôpital's rule

(a) $\lim_{x\to 0} \frac{\sin(x)}{x}$

(b) $\lim_{x\to\infty} \frac{\ln(x)}{x}$

(a) $\lim_{x\to 0} \frac{\sin(x)}{x} = \frac{0}{0}$

L'H

$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\cos(x)}{1} = 1.$ ✓

(b) $\lim_{x\to\infty} \frac{\ln(x)}{x} = \frac{\infty}{\infty}$

L'H

$\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$

6. (15 Points) Find an inverse function for the following functions and state their domains.

(a) $f(x) = e^x$

(b) $h(x) = e^{2x} + 1$

(c) $g(x) = x^2 + 4x + 4$ when $x \leq -2$. (It might be helpful to graph this function)

(a) $f^{-1}(x) = \ln(x)$, domain $= (0, \infty)$

(b) $y = e^{2x} + 1 \Rightarrow y - 1 = e^{2x}$

$\Rightarrow \ln(y-1) = 2x$

$\Rightarrow x = \dfrac{\ln(y-1)}{2}$

$h^{-1}(y) = \dfrac{\ln(y-1)}{2}$, domain $= (1, \infty)$

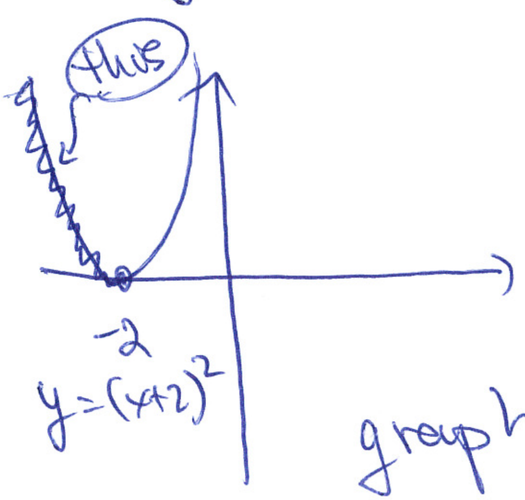
(c) $g(x) = x^2 + 4x + 4$

$= (x+2)^2$.

$y = (x+2)^2$, $\pm\sqrt{y} = x + 2$

$\Rightarrow x = -2 \pm \sqrt{y}$

take negative because of domain



graph

$\boxed{g^{-1}(y) = -2 - \sqrt{y}}$ domain $(0, \infty)$

5. (15 points) Find the following limits. (You can just state the answer if you know it.)

(a) $\lim_{x\to\infty} e^{-x}\cos(x)$

(b) $\lim_{x\to\infty} \tan^{-1}(x)$

(c) $\lim_{x\to\infty} \frac{2e^x - e^{-x}}{e^x + e^{-x}}$

(a) $\lim_{x\to\infty} e^{-x}\cos(x) = 0$

(b) $\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}$

(c) $\lim_{x\to\infty} \frac{2e^x - e^{-x}}{e^x + e^{-x}} = 2$

7. (15 Points) Let $f^{-1}(t)$ be the inverse function of $f(s) = s + e^s$. Find the line tangent to the graph of $f^{-1}(t)$ at $t = 1$. (Hint: don't try to compute the inverse directly like in problem 6)

Use the inverse function theorem

$$(f^{-1})'(t) = \frac{1}{f'(f^{-1}(t))}$$

Calculate Point

$$f(0) = 0 + e^0 = 1 \implies \underbrace{0 = f^{-1}(1)}_{\text{point}}$$

Calculate Slope

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{e^0} = \underbrace{1}_{\text{slope}}$$

Note:

$$\boxed{f'(s) = \frac{d}{ds}[s + e^s] = e^s}$$

Point-Slope Eqn of Line:

$$(s - s_0) = m(t - t_0)$$

$$\implies (s - 0) = 1(t - 1)$$

$$\implies \boxed{s = t - 1}$$ ← equation of tangent line

8. (15 Points) Derive the formula for the derivative of $\sin^{-1}(x)$ where the domain of $\sin(x)$ is taken to be $[-\pi/2, \pi/2]$.

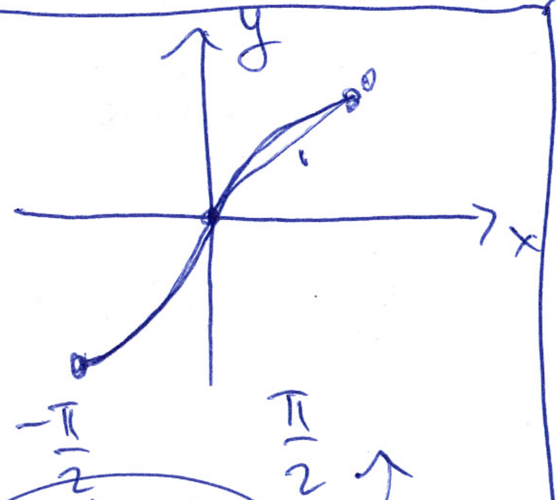
$\cancel{\sin(\sin^{-1}(x)) =}$

$$\sin^{-1}(x) = y \Rightarrow x = \sin(y)$$

$$\Rightarrow 1 = \cos(y)\, y'$$

$$\Rightarrow \frac{1}{\cos(y)} = y'$$

$$\sin(y)^2 + \cos(y)^2 = 1 \Rightarrow \cos(y) = (+)\sqrt{1-\sin(y)^2}$$



graph of $\sin(x)$

~~[illegible]~~

(slope is always positive on domain)

$$y' = \frac{1}{\sqrt{1-\sin(y)^2}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

So we have shown

$$\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}.$$