KEY

Math 163 — Test 02

Thursday, October 18th 2012

Instructions Remember to show all your work so you can get partial credit. You shouldn't need a calculator on this test but you can use one. Please leave answers in their exact form. Try not to overthink the problems too much.

1. (15 Points) Compute the following integrals
   (a) $\int xe^x dx$.
   (b) $\int x^2 \sin(x) dx$.
   (c) $\int \ln(x) dx$

a) $\int xe^x dx = uv - \int v\,du = xe^x - \int e^x dx$

$u = x$, $du = dx$

$dv = e^x dx$, $v = e^x$

$= xe^x - e^x + C$

b) $\int x^2 \sin(x)\, dx = -x^2\cos(x) + \int \cos(x)\cdot 2x\, dx$

$u = x^2$, $dv = \sin(x)dx$

$du = 2x dx$, $v = -\cos(x)$

$u = x$ $dv = \cos(x)dx$

$du = dx$ $v = \sin(x)$

$= -x^2\cos(x) + 2\left[\sin(x)\, x - \int \sin(x) dx\right]$

$= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$

c) $\int \ln(x) dx = x\ln(x) - x + C$.

2. (10 Points) Compute the following indefinite integrals

(a) $\int \frac{1}{x^2-1} dx$

(b) $\int \frac{1}{x^3+x} dx$

a)

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}, \quad 1 = A(x+1) + B(x-1)$$

$x=1$: $1 = A \cdot 2 \Rightarrow A = 1/2$

$x=-1$: $1 = 0 + B(-2)$

$\Rightarrow B = -1/2$.

$$\frac{1}{x^2-1} = \frac{1}{2}\frac{1}{x-1} + \frac{-1}{2}\frac{1}{x+1}$$

$$\int \frac{1}{x^2-1} dx = \frac{1}{2}\int \frac{1}{x-1} dx - \frac{1}{2}\int \frac{1}{x+1} dx$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

b)

$$\frac{1}{x^3+x} = \frac{1}{x} \cdot \frac{1}{x^2+1} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow A(x^2+1) + (Bx+C)x = 1$$

$x=0$: $A(0+1) = 1 \Rightarrow A = 1$

$x=1$: $2A + B + C = 1$

$x=-1$: $2A + (-B+C) = 1$

$$\Rightarrow \begin{cases} B+C = -1 \\ B-C = -1 \end{cases}$$

$\Rightarrow 2B = -2 \Rightarrow B = -1$

$\Rightarrow C = 0$.

$$\frac{1}{x^3+x} = \frac{1}{x} - \frac{x}{x^2+1} \Rightarrow \int \frac{1}{x^3+x} dx = \ln|x| - \int \frac{x}{x^2+1} dx =$$

$u = x^2+1$, $du = 2x\,dx$

$$= \ln|x| - \frac{1}{2}\int \frac{1}{u} du = \ln|x| - \frac{1}{2}\ln|x^2+1| + C.$$

8. (For Respect/ If you want to see something cool)

(a) Find the constants λ such that functions of the form $y(t) = e^{\lambda t}$ give a solution to the differential equation

$$y'' = -y.$$

(b) It turns out that $\lambda = \pm i$ in the previous problem. It also turns out that every solution of the equation $y'' = -y$ are of the form $y(t) = Ae^{it} + Be^{-it}$. But what the hell?! What about the solution $y(t) = \cos(t)$ that we did in the previous problem??

Here is the exercise you should do: Using Euler's Formula $e^{it} = \cos(t) + i\sin(t)$ find the constants A and B such that $\cos(t) = Ae^{it} + Be^{-it}$. (Hint: do the computation $e^{it} + e^{-it}$ then adjust it slightly)

a) $y(t) = e^{\lambda t}$

$$\begin{cases} y'(t) = \lambda e^{\lambda t} \\ y''(t) = \lambda^2 e^{\lambda t} \end{cases}$$

$y'' + y = 0$

$\Rightarrow e^{\lambda t} + \lambda^2 e^{\lambda t} = 0$

$\Rightarrow e^{\lambda t}(1 + \lambda^2) = 0$

$\Rightarrow 1 + \lambda^2 = 0 \Rightarrow \lambda = \pm i$

b)

$$\cos(t) = \frac{1}{2}\left(e^{it} + e^{-it}\right)$$

<u>check this:</u>

$$e^{it} + e^{-it} = e^{it} + e^{i(-t)}$$

$$= (\cos(t) + i\sin(t)) + (\cos(-t) + i\sin(-t))$$

$$= \cos(t) + i\sin(t) - i\sin(t) + \cos(t)$$

$$= 2\cos(t)$$

$$\Rightarrow \frac{e^{it} + e^{-it}}{2} = \cos(t). //$$

7. (20 Points)

(a) (10 Points) Verify that $y(t) = \cos(t)$ is a solution of the differential equation

$$\frac{d^2y}{dt^2} = -y.$$

(b) (10 Points) Verify that $y(t) = e^{it}$ is a solution of the differential equation

$$\frac{d^2y}{dt^2} = -y.$$

Here, $i = \sqrt{-1}$ and it is the imaginary number which satisfies $i^2 = -1$. You treat it like a constant.

a) $y'(t) = -\sin(t)$

$y''(t) = -\cos(t)$.

$\frac{d^2y}{dt^2} + y = -\cos(t) + \cos(t) = 0$. ✓

b) $y'(t) = ie^{it}$

$y''(t) = -e^{it}$

$\frac{d^2y}{dt^2} + y = -e^{it} + e^{it} = 0$. ✓

6. (10 Point) Compute the following integral $\int e^x \sin(x) dx$.

$$\int e^x \sin(x)\,dx$$

$\cancel{u = \sin(x)}$ $u = e^x$, $dv = \sin(x)dx$

$\cancel{du}$ $du = e^x dx$, $v = -\cos(x)$

$$\int e^x \sin(x)\,dx = -e^x \cos(x) + \int e^x \cos(x)\,dx$$

$u = e^x$, $dv = \cos(x)dx$

$du = e^x dx$, $v = \sin(x)$

$$= -e^x \cos(x) + \left[e^x \sin(x) - \int e^x \sin(x)\,dx \right]$$

$$\Rightarrow 2\int e^x \sin(x)\,dx = -e^x \cos(x) + e^x \sin(x)$$

$$\Rightarrow \boxed{\int e^x \sin(x)\,dx = \frac{1}{2}\left[-e^x \cos(x) + e^x \sin(x)\right] + C}$$

5. (15 Points) Solve the following initial value problem

$$\begin{aligned} \frac{dx}{dt} &= x^2 t, \\ x(0) &= 1. \end{aligned}$$

$$\frac{dx}{dt} = x^2 t \Rightarrow \int \frac{dx}{x^2} = \int t\,dt$$

$$\Rightarrow \frac{-1}{x} = \frac{t^2}{2} + C.$$

$$\frac{-1}{1} = \frac{-1}{x(0)} = 0 + C \Rightarrow C = -1.$$

$$\Rightarrow \frac{-1}{x(t)} = \frac{t^2}{2} - 1$$

$$\Rightarrow \boxed{x(t) = \frac{-1}{t^2/2 - 1}}$$

4. (15 Points) Evaluate the following improper integrals

(a) $\int_0^\infty te^{-t}dt$

(b) $\int_0^1 \frac{1}{\sqrt{x}}dx$

(c) $\int_{-\infty}^\infty \frac{dx}{1+x^2}$.

a) $\int_0^\infty te^{-t}dt = -te^{-t}\Big|_0^\infty + \int_0^\infty e^{-t}dt$

$u = t,\ dv = e^{-t}dt$

$du = dt,\ v = -e^{-t}$

$= 0 + \int_0^\infty \frac{d}{dt}\left[-e^{-t}\right]dt$

$= -e^{-t}\Big|_0^\infty = 0 + 1$

$= 1.$

b) $\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 x^{-1/2}dx$

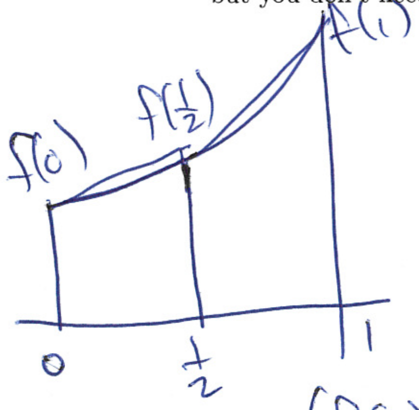
$= \lim_{\varepsilon \to 0^+} \frac{x^{-1/2+1}}{-1/2+1}\Big|_\varepsilon^1$

$= \lim_{\varepsilon \to 0^+}\left(2\sqrt{1} - 2\sqrt{\varepsilon}\right) = 2.$

c) $\int_{-\infty}^\infty \frac{dx}{1+x^2} = \tan^{-1}(x)\Big|_{-\infty}^\infty = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$

3. (15 Points) Estimate the definite integral using the trapezoidal rule with two intervals. The notation they used in the book and online for this is T_2. It would be nice if you did this with a calculator but you don't need to.

$$\int_0^1 e^{x^2}\, dx.$$



$$T_2 = \left(\frac{f(0)+f(\frac{1}{2})}{2}\right)\frac{1}{2} + \left(\frac{f(\frac{1}{2})+f(1)}{2}\right)\frac{1}{2}$$

$$= \left(\frac{1+e^{(1/2)^2}}{2}\right)\frac{1}{2} + \left(\frac{e^{(1/2)^2}+e}{2}\right)\frac{1}{2}$$

$$= \frac{1}{4} + \frac{1}{2}e^{(1/2)^2} + \frac{e}{4}.$$

≈ 1.5716