Math 31B — Homework 08

1. Show that the series
$$\sum_{n\geq 1}\frac{1}{n}$$
diverges using the definition of convergence.

2. Find the series and the radius of convergence:
   (a) $\log_b(x)$ at $x_0 = 1$
   (b) $\log_b(x+1)$ at $x_0 = 0$
   (c) $\log_b(x+1)$ at $x_0 = 2$.

   (Make sure to test endpoints)

3. Determine whether the following series converge or diverge
   (a) $\sum_{n\geq 1}\frac{1}{n^2+n}$
   (b) $\sum_{n\geq 1}\frac{\ln(n)}{n}$
   (c) $\sum_{n=2}^{\infty}\frac{1}{\ln(n)^2}$

4. Determine whether the following sequence or converge or diverge as $n \to \infty$
   (a) $a_n = (-1)^n$
   (b) $b_n = (-1)^n/2^n$
   (c) $c_n = 1 + (-1)^n/2^n$

5. The Bernoulli numbers are defined by the formula
$$\frac{x}{e^x - 1} = \sum_{n\geq 0} B_n \frac{x^n}{n!}.$$
   Using long division find the first four Bernoulli numbers. (One the second midterm I was intending you to find these using the Taylor Series formula together with L'Hôpital's rule).

6. (a) Find the Taylor series for $\tan^{-1}(x)$. (Hint: integrate $1/(1+x^2)$).
   (b) Find a series for $\pi/4$ using that $\tan^{-1}(1) = \pi/4$.
   (c) How many terms of this approximation above does one need to take to approximate $\pi/4$ accurate to 6 decimal places?

7. Find the radius of convergence of the following Taylor series
   (a) $\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}x^{2n}$
   (b) $\sum_{j=1}^{\infty}(\frac{-2n}{n+1})^{5n}x^n$

8. Find the interval of convergence for $\frac{1}{1-2x}$ and $\ln(\frac{1}{1-x})$ (make sure to test the endpoints).

9. Consider the series
$$f(x) = \sum_{n=0}^{\infty}\frac{\sin(3^n \pi x)}{2^n}.$$
   (this series is an example of a fractal).

(a) Show that the series converges for every x

(b) Show that the series for $f'(x)$ is not absolutely convergent.

(c) Show that $f'(0)$ is divergent.

(d) Suppose that a and b are positive numbers bigger than one. The following two exercises are to remind you about how to manipulate graphs.

i. Explain why the graph of $g(ax) = by$ is just the graph of $g(x) = y$ where the x axis is scaled by $1/a$ and the y-axis is scaled by $1/b$. (Hint: Consider the graph of $g(\tilde{x}) = \tilde{y}$ and let $\tilde{x} = ax$ and $\tilde{y} = by$. To obtain x we shrink $\tilde{x}$ by a and to obtain y we shrink $\tilde{y}$ by b).

ii. Explain why the graph of $g(x - a) = y - b$ is just the graph of $g(x) = y$ where the x-axis shifted the the right by a and the y-axis is shifted up by b.

(e) The following exercises show the "self-similarity" this fractal

i. Show that $f(x + 1) = -f(x)$. (Show that $f(x)$ is periodic with period 2)

ii. Explain why shifting the graph to the left by 1 is the same as reflecting it across the y-axis.

iii. Show that $f(3x)/2 = f(x) - \sin(\pi x)$.

iv. Explain why shrinking the graph $y = f(x)$ by $1/3$ on the x-axis and by $1/2$ on the y-axis is the same as shifting the whole graph down by $\sin(\pi x)$.

(f) Draw a graph of $f(x)$.

10. Find the value of the following continued fraction

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

(Hint if x is equal to the continued fraction you have $x = 1 + 1/x$)

11. Let $g(x)$ the the inverse function of $f(x) = x + x^5$. Find the first three terms of the Taylor series for $g(x)$ centered at $x = 0$.