TEST 2 — Math 264 — Fall 2010

November 2, 2010

Show all of your work. Remember to use English sentences where necessary. If you don't know how to do a problem explain what you do and don't know to get as many points as possible.

1. Compute $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for the following functions

   (a) $u(x, y) = e^{xy}$.

   (b) $u(x, y) = \sin(xy)y$.

(a) $u_x = ye^{xy}$

$u_y = xe^{xy}$

+5

(b) $u_x = \cancel{y\cos(xy)x}$ $y^2\cos(xy)$

+5

$u_y = \cancel{\cos(xy)y^2 + \sin(xy)y.}$

$\sin(xy) + xy\cos(y)$

2. Show that the function $u(x, t) = f(x - ct) + g(x + ct)$ satisfies the wave equation $u_{tt} - c^2u_{xx} = 0$.

$u_{xx} = f''(x-ct) + g''(x+ct)$

+5

$u_{tt} = c^2f''(x-ct) + c^2g''(x+ct)$

$\Rightarrow u_{tt} - c^2u_{xx} = c^2[f''(x-ct) + g''(x+ct)]$ +5

$- c^2[f''(x-ct) + g''(x+ct)]$

$= 0.$ //

two variables

3. (a) How can you show that a limit of a function in TWO doesn't exist?

(b) Show that $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^3}$ doesn't exist.

(a) Approach the point along two different curves & get two different values. +4

(b) Let $\vec{r}_1(t) = (t,t)$, $\vec{r}_2(t) = (t,0)$ &

$f(x,y) = \frac{x^2y}{x^3+y^3}$.

+6

$f(\vec{r}_1(t)) = \frac{t^3}{t^3+t^3} = \frac{1}{2} \to \frac{1}{2}$ as $t \to 0$.

$f(\vec{r}_2(t)) = \frac{t^2(0)}{t^3+(0)^3} = 0 \to 0$ as $t \to 0$

Since $1/2 \neq 0$ the limit doesn't exist.

4. Let $f(x,y)$ be a function of two variables. Using the chain rule show that $f_r = \frac{f_x x + f_y y}{\sqrt{x^2+y^2}}$; where $x = r\cos(\theta)$ and $y = r\sin(\theta)$

$f_r = f_x x_r + f_y y_r$

$= f_x \frac{x}{\sqrt{x^2+y^2}} + f_y \frac{y}{\sqrt{x^2+y^2}}$

+5

$= \frac{f_x x + f_y y}{\sqrt{x^2+y^2}}$. +5

SIDE WORK:

$x_r = \cos(\theta) = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$,

$y_r = \sin(\theta) = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$.

5. Find the plane tangent to the graph of the following functions at the specified point.

(a) $f(x, y) = e^{xy} + x^2$, $P = (0, 1, 1)$.

(b) $g(x, y) = x^3 + y^3$, $Q = (-1, 1, 0)$.

(a) $f_x = ye^{xy} + 2x \Rightarrow f_x(0,1) = 1$

$f_y = xe^{xy} \Rightarrow f_y(0,1) = 0$ +5

$z = f(0,1) + \nabla f(0,1) \cdot (x-0, y-1)$

$\Rightarrow \boxed{z = 1 + (1,0) \cdot (x, y-1)}$ or $z = x+1$

(b) $g_x = 3x^2 \Rightarrow g_x(-1,1) = 3$

$g_y = 3y^2$ $\quad g_y(-1,1) = 3$ +5

$z = g(-1,1) + \nabla g(-1,1) \cdot (x+1, y-1)$

$\Rightarrow \boxed{z = 0 + (3,3) \cdot (x+1, y-1)}$

or $z = 3(x+1) + 3(y-1)$

6. Find the plane tangent to the surface at the specified point.

(a) $x^4 + y^2 + z^2 = 1$, $P = (1, 0, 0)$

(b) $z - x^3 - y^3 = 0$, $Q = (-1, 1, 0)$. (Hint: compare to part *b* of the previous problem)

(a) Let $g(x,y,z) = x^4 + y^2 + z^2$. $g_x = 4x^3$, $g_y = 2y$

$g_z = 2z$, $\Rightarrow \nabla g(1,0,0) = (4,0,0)$.

+5 $\quad \nabla g(1,0,0) \cdot (x-1, y-0, z-0) = 0$ (formula for tan plane)

$\Rightarrow$ $\boxed{4(x-1) = 0 \text{ is the tangent plane}}$

(b) setting $f(x,y) = x^3 + y^3$ and finding the tangent plane of the graph of this function is exactly the same problem, so

+5 $\quad \boxed{z = 3(x+1) + 3(y-1).}$

+4

7. Find the critical points of $f(x, y) = -(x^2 - 1)^2 - y^2$ and classify them.

CRIT PTS: $\nabla f = 0 \Rightarrow \begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} -2(x^2-1)(2x) = 0 \\ -2y = 0 \end{cases}$

$\Rightarrow x = \pm 1$ or $x = 0$ with $y = 0$. So the critical points are $(1,0)$, $(-1,0)$ & $(0,0)$.

ANALYSIS OF CRIT PTS: +3

$$H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -4(3x^2-1) & 0 \\ 0 & -2 \end{bmatrix}$$

+3

(x,y):	$\det H(x_0,y_0)$	$f_{xx}(x_0,y_0)$	conclusion
$(1,0)$	$16>0$	-8	concave down $\Rightarrow$ max
$(-1,0)$	$16>0$	-8	concave down $\Rightarrow$ max
$(0,0)$	$-8<0$	not needed	saddle

8. Find the maximal volume for a rectangle inscribed in the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$.

fn you want to optimize:

+6 $V(x,y,z) = 8xyz$, $x \geq 0, y \geq 0, z \geq 0$

(solid is symmetric in xy, yz & xz planes)

Constraint:

$x^2 + 2y^2 + 3z^2 = 1$

$$\begin{cases} \nabla V = \lambda \nabla g \\ x^2 + 2y^2 + 3z^2 = 1 \end{cases} \Rightarrow \begin{cases} 8yz = \lambda 2x \\ 8xz = \lambda 4y \\ 8xy = \lambda 6z \\ x^2 + 2y^2 + 3z^2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} 8xyz = \lambda 2x^2 \\ 8xyz = \lambda 4y^2 \\ 8xyz = \lambda 6z^2 \\ x^2 + 2y^2 + 3z^2 = 1 \end{cases} \Rightarrow 3\left(\frac{8xyz}{2\lambda}\right) = 1 \Rightarrow 8xyz = \boxed{\frac{2\lambda}{3}}$$

for whatever volume λ is.

$\Rightarrow \lambda 2x^2 = \frac{2\lambda}{3} \Rightarrow \lambda\left(2x^2 - \frac{2}{3}\right) = 0$, $\lambda \neq 0 \Rightarrow 2x^2 - \frac{2}{3} = 0$

$\Rightarrow x = \pm 1/\sqrt{3}$, Similarly $4y^2 - \frac{2}{3} = 0$ & $6z^2 - \frac{2}{3} = 0$

$\Rightarrow\ y = +\sqrt{\frac{1}{6}}\ ,\ z = +\sqrt{\frac{1}{9}} = \frac{1}{3}.$ +4

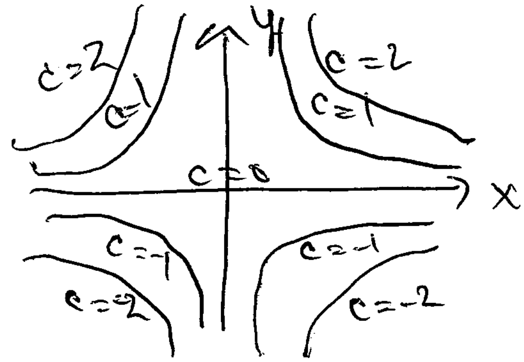
$8xyz = 8\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{3}\right) = \frac{8}{9\sqrt{2}}.$ //

9. (a) Plot the level curves of the function $f(x, y) = x^2 y$.

(b) What do the level surfaces of the function $g(x, y, z) = x^2 + 4y^2 + z^2$ look like? (A one sentence answer is acceptable)

a) $x^2 y = c \Rightarrow y = \frac{c}{x^2}$.

+5



(b) They look like ellipsoids of various volumes.

+5

10. Suppose that $f(x, y)$ is a differentiable function and ~~C is some C is a real number.~~ $c \in \mathbb{R}$. If $\vec{r}(t)$ is differentiable and parametrizes the level set $\{(x, y) : f(x, y) = C\}$. Show that for all t in the domain, ~~of~~ $\vec{r}'(t)$ and $(\nabla f)(\vec{r}(t))$ are perpendicular.

$$f(\vec{r}(t)) = c$$

$$\Rightarrow \frac{d}{dt}\left[f(\vec{r}(t))\right] = 0$$

+5

(since the derivative of a constant is zero)

$$\Rightarrow \underbrace{Df(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0}_{+5}$$

(By the chain rule)

11. (EXTRA CREDIT)

(a) For $\vec{r}: \mathbb{R}^2 \to \mathbb{R}^2$ is given by $\vec{r}(s,t) = (x(s,t), y(s,t))$ what is the Jacobian Matrix of $\vec{r}$?

(b) How does is provide a way to view the chain rule for several variables as a "chain rule". (Write the pair of equations that we call the chain rule as a single matrix equation involving the jacobian).

(c) Show that the Hessian Matrix is the Jacobian of the gradient.

+3

(a)

$$(D\vec{r})(s,t) = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}$$

(b) If $g(s,t) = f(x(s,t), y(s,t))$ we have

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

(+4

this can be written as a matrix equation

$$\begin{aligned}(\nabla g)(s,t) &= \left(\frac{\partial g}{\partial s}, \frac{\partial g}{\partial t}\right) \\ &= \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}, \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\right) \\ &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix} \\ &= (\nabla f)(x(s,t), y(s,t))\,(D\vec{r})(s,t)\end{aligned}$$

Matrix Mult

So $$\boxed{\nabla g(s,t) = \nabla f(x(s,t), y(s,t))\,(D\vec{r})(s,t)}$$

(c) The Hessian is $H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$.

The gradient is $\nabla f = (f_x, f_y)$. If we let $u = f_x$ & $v = f_y$ we have

$$D(\nabla f) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

+3

$$= \begin{bmatrix} \frac{\partial}{\partial x}[f_x] & \frac{\partial}{\partial y}[f_x] \\ \frac{\partial}{\partial x}[f_y] & \frac{\partial}{\partial y}[f_y] \end{bmatrix}$$

$$= \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = H(x,y).$$ //

Other Extra Credit:

1. Prove $P \neq NP$ +1
2. Prove Riemann Hypoth +1
3. Prove Mass Gap for Yang-Mills +1
4. Prove Birch & Swinnerton-Dyer conjecture +1
5. Show existence of Solution to Navier-Stokes equations +1
6. Hodge Conjecture +1
7. Poincare Conjecture +1