A Quick Peek at Complex Differentiable Functions

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October 2, 2009

A function of a complex variable is a function $f : D \to \mathbb{C}$ where D is some domain of the complex numbers where the function is defined.

- Polynomials like $f(z) = z^2 + 2$ can be considered complex valued functions.
- The exponential function $\exp(z) = e^z$ can be considered as a complex valued function.

As stated before the complex numbers $\mathbb{C}$ consist of numbers of the form $z = x + iy$. That is, they can be put into correspondence with pairs of real numbers (x, y):

$$x + iy \leftrightarrow (x, y).$$

The only difference between $\mathbb{C}$ and $\mathbb{R}^2$ is that $\mathbb{C}$ has a multiplication.

A useful thing to notice is that every complex valued function can be broken up into it's real an imaginary parts:

$$f(z) = f(x + iy) = u(x, y) + iv(x, y),$$

where $u(x, y)$ and $v(x, y)$ are now functions which take in two real inputs and output one real number. For example, if $f(z) = z^2$ then we have

$$z^2 = (x + iy)^2 = x^2 - y^2 + i(2xy),$$

here $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Here is how we do it for the exponential function $\exp(z)$:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y)) = e^x \cos(y) + ie^x \sin(y).$$

Here we have $u(x, y) = e^x \cos(y)$ and $v(x, y) = e^x \sin(y)$. In breaking up the exponential function we used Euler's Formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Definition Let D be a subset of the complex numbers. A function $f : D \to \mathbb{C}$ is said to be analytic provided that for every $z \in D$ the limit

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z) \tag{1}$$

exists. **Here h is taken to be a complex number and taking $h \to 0$ in ANY way needs to give the same number $f'(z)$!**

The statement about a function is saying something interesting about all the directional derivatives. If we take $h = t$, (t is real) and let $t \to 0$ we are essentially taking a partial derivative with respect to x, or the directional derivative in the direction of x. If we take $h = it$ and let $t \to 0$ then we are essentially taking a partial derivative with respect to y.

Let's be more explicit. When letting $h = t$ we have,

$$\begin{aligned} f'(z) = \lim_{t\to 0} \frac{f(z+t) - f(z)}{t} &= \lim_{t\to 0} \frac{f(x+t+iy) - f(x+iy)}{t} \\ &= \lim_{t\to 0} \frac{[u(x+t,y) + iv(x+t,y)] - [u(x,y) - iv(x,y)]}{t} \\ &= \lim_{t\to 0} \left(\frac{u(x+t,y) - u(x,y)}{t} + i\frac{v(x+t,y) - v(x,y)}{t} \right) \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \end{aligned}$$

When letting $h = it$ we have

$$
\begin{aligned}
f'(z) = \lim_{t\to 0} \frac{f(z+it) - f(z)}{it} &= \lim_{t\to 0} -i \frac{f(x+i(y+t)) - f(x+iy)}{t} \\
&= \lim_{t\to 0} -i \frac{[u(x,y+t) + iv(x,y+t)] - [u(x,y) - iv(x,y)]}{t} \\
&= -i \lim_{t\to 0} \left(\frac{u(x,y+t) - u(x,y)}{t} + i \frac{v(x,y+t) - v(x,y)}{t} \right) \\
&= -i \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \\
&= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.
\end{aligned}
$$

On the second equality step we used the fact that $1/i = -i$. This might have some people worried since we have two different expression for $f'(z)$! But since we didn't do anything wrong they must be equal:

Theorem 0.1 (Cauchy-Riemann Equations) *If $f(z) = u(x,y) + iv(x,y)$ is complex differentiable then*

$$
f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.
$$

*By equating real an imaginary parts of these two expressions we get the **Cauchy-Riemann Equations**:*

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}.
\end{aligned}
$$

Let's look at this for some examples. For $f(z) = z^2$ we have $u = x^2 - y^2$ and $v = 2xy$,

$$
\begin{aligned}
u_x &= 2x, \\
u_y &= -2y, \\
v_x &= 2x, \\
v_y &= 2y.
\end{aligned}
$$

One can check that the equations work: $u_x = v_y$ and $u_y = -v_x$.

Now consider the exponential function $\exp(z) = e^z$. Here we have $u = e^x \cos(y)$ and $v = e^x \sin(y)$.

$$
\begin{aligned}
u_x &= e^x \cos(y), \\
u_y &= -e^x \sin(y), \\
v_x &= e^x \sin(y), \\
v_y &= e^x \cos(y).
\end{aligned}
$$

The component functions u and v of a are so important that we give them a name.

Definition If $u(x,y)$ is the component function of a complex differentiable function $f(z)$ i.e. $f(z) = u(x,y) + iv(x,y)$ then we say that $u(x,y)$ is **harmonic** and call $v(x,y)$ the **harmonic conjugate** of $u(x,y)$.