Complex Numbers and Quaternions for Calc III

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Abstract

An introduction to complex numbers and quaternions

1 Introduction

I'm going to review some facts about the complex numbers and then generalize them to quaternions. The quaternions historically formed the basis for the cross product in vector analysis.

Vector Spaces over the real numbers are just required two operations:

1. Vector Addition: a way of adding two vector quantities say $\vec{v}$ and $\vec{w}$ and getting a new vector quatities that satisfies the usual rules of an addition (commutativity, associativity, etc.)

2. Scalar Multiplication: Scalar a way of taking a real number and c multiplying it times a vector $\vec{v}$ and getting out a new vector $c\vec{v}$ that lies in our vector space.

It turns out that the set of complex numbers (denoted by $\mathbb{C}$) form a vector space. As a vector space $\mathbb{C}$ looks essentially like the cartesian plane (denoted by $\mathbb{R}2$). It turns out that $\mathbb{C}$ has *more* structure then just these operations. The complex numbers give a way of *multiplying* two two-dimensional vectors getting a new vectors out... so there is a *vector multiplication*[1]. It turns out that this multiplication has a nice geometric interpretation as rotations and expansions.

A the multiplication that exists for the complex numbers is extremely rare. In fact there are only three spaces of it's kind.[2] You know two of them: The set of real numbers and the set of quaternion. The third space is the set of quaternions and is actually four dimensional. Historically, when you multiply the cross product comes from a component of quatenion multiplcation.

2 Two Ways of Looking at Complex Numbers

Abstractly we could define a the complex numbers as being ordered pairs of real numbers (x, y) with the following multiplication operation:

$$(x_1, y_1) * (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) \tag{1}$$

[1]Satisfying a set of axioms including things like commutivity, associativity and the existence multiplicative inverses.

[2]To be technical, they are the only "Finite dimensional Banach Algebras" over the real numbers. This fancy words which essentially means things you can do sequences and series with.

So if you handed me two vectors say, $(1,1)$ and $(2,-1)$ and asked me to multiply them I would just use the rule:

$$\begin{aligned}(1,1)*(2,-1) &= (1\cdot 2-1\cdot(-1), 1\cdot(-1)+1\cdot 2)\\ &= (2+1,-1+2)\\ &= (3,1).\end{aligned}$$

Using these rules one could check it's properties: that this multiplication is commutative, associative, and that is is obeys distributive properties with respect to the vector addition

$$(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2).$$

For example proving the left distributive property would amount to proving the equality

$$(x_1,y_1)*((x_2,y_2)+(x_3,y_3))=(x_1,y_1)*(x_2,y_2)+(x_1,y_1)*(x_3,y_3),$$

from the rules we have defined above.

Here is another way to define the complex numbers: A **complex number** is a number of the form $x+iy$ where x and y are real numbers. Here, i is the square root of negative one $i=\sqrt{-1}$. It satisfies the identity

$$i^2=-1.$$

Note that when we write complex numbers in this way, the rule for complex multiplication takes the form

$$\begin{aligned}(x_1+iy_1)(x_2+iy_2) &= x_1(x_2+iy_2)+iy_1(x_2+iy_2)\\ &= x_1x_2+ix_1y_2+iy_1x_2-y_1y_2\\ &= (x_1x_2-y_1y_2)+i(x_1y_2+y_1x_2)\end{aligned}$$

and that under the identification

$$(x,y)\leftrightarrow x+iy$$

our rule crazy looking rule (1) is just the rule for complex multiplication.

For example

$$\begin{aligned}(1+i)(2-i) &= (1+i)2+(1+i)(-i)\\ &= (2+2i)+(-i+1)\\ &= 3+i.\end{aligned}$$

The same sort of things is going on with the cross product of two vectors and quaternion multiplication.

3 Geometry of Complex Numbers

The set of complex numbers is denoted by $\mathbb{C}$ and as a reminder,

$$\mathbb{C}=\{x+iy|x,y\in\mathbb{R}\}.$$

The set of complex numbers form a vector space over $\mathbb{R}$.

Every complex number $z=x+iy$ consists essentially of the data of two real numbers and we can in fact put the set of complex numbers in correspondence with points in the cartesian plane $\mathbb{R}^2$. As said before, the correspondence is $x+iy=(x,y)$.

On $\mathbb{R}^2$ we have **norm/length/magnitude** of vectors defined by

$$|(x,y)|=\sqrt{x^2+y^2}.$$

So for example the length of $(1,2)$ is the vector $\sqrt{1+4}=\sqrt{5}$. As stated before, the complex numbers are essentially the same as the points in $\mathbb{R}^2$ with the additional structure of multiplication. So we defined the length of a complex number in the exact same way as we defined the length of a two dimensional vector: If $z=x+iy$ then $|z|=\sqrt{x^2+y^2}$.

Recall that **unit vectors** are ones whose magnitude are one.

Example • The complex number $1/\sqrt{2}+i/\sqrt{2}$ is a unit complex number. This is because

$$|\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}| = \sqrt{\frac{1}{2}+\frac{1}{2}} = 1.$$

• The complex number $1/2+i\sqrt{3}/2$ is a unit complex number. This is because

$$|\frac{1}{2}+i\frac{\sqrt{3}}{2}| = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1.$$

These examples may look familiar. Note that $1/\sqrt{2}+i/\sqrt{2} = \cos(\pi/4)+i\sin(\pi/4)$ and $1/2+i\sqrt{3}/2 = \cos(\pi/6)+i\sin(\pi/6)$. In fact all the unit complex numbers will be of the form $z = \cos(\theta)+i\sin(\theta)$. This is because $|z| = 1$, means that $|z|^2 = 1$. But $|z|^2 = x^2+y^2$ so this means that if $z = x+iy$ is a unit vector the x and y must satisfy $x^2+y^2 = 1$.

We will show briefly that unit complex numbers are closed under multiplication. That is, if z_1 and z_2 are unit complex numbers then z_1z_2 is also a unit complex number. Let first verify this with an example.

Example We already showed that $z_1 = 1/\sqrt{2}+i/\sqrt{2}$, and $z_2 = 1/2+i\sqrt{3}/2$ are complex numbers. We now can multiply these together then check that z_1z_2 is in fact a unit complex number.

In addition to a multiplication, the set of complex numbers has an operation called **complex conjugation**. Here is the operation:

$$\overline{x+iy} = x-iy.$$

Example The complex conjugate of the number $1+i$ is $\overline{1+i} = 1-i$.

The operation of complex conjugation splits up over sums and products of complex numbers as well as interacts with

Proposition 3.1 *1. We can write the norm squared of a complex number as a product of a complex number and it's conjugate*

$$|z|^2 = z\bar{z}$$

.

2. Complex Conjugation breaks up over sums:

$$\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$$

3. Complex Conjugation breaks up over products:

$$\overline{z_1z_2} = \bar{z}_1\bar{z}_2.$$

Proof 1. If $z = x+iy$ we can write its norm squared in a special way:

$$\begin{aligned} |z|2 &= x2+y2 \\ &= (x-iy)(x+iy) \\ &= \bar{z}z. \end{aligned}$$

which shows what we wanted.

2. To see this, first let $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$,

$$\begin{aligned} \overline{z_1+z_2} &= \overline{x_1+x_2+i(y_1+y_2)} \\ &= x_1+x_2-i(y_1+y_2) \\ &= x_1-iy_1+x_2-iy_2 \\ &= \bar{z}_1+\bar{z}_2 \end{aligned}$$

3.

$$\begin{aligned} \overline{z_1 z_2} &= \overline{x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)} \\ &= x_1x_2 - y_1y_2 - i(x_1y_2 + x_2y_1) \\ &= (x_1 - iy_1)(x_2 - iy_2) \\ &= \bar{z}_1\bar{z}_2 \end{aligned}$$

Corollary 3.2 (Norm of the Product is the Product of the Norms.) *For any two complex numbers* $|z_1z_2| = |z_1| \cdot |z_2|$.

Proof To prove this we just apply all the things that we learned before:

$$\begin{aligned} |z_1z_2|^2 &= \overline{z_1z_2}z_1z_2 \\ &= \bar{z}_1\bar{z}_2z_1z_2 \\ &= \bar{z}_1z_1\bar{z}_2z_2 \\ &= |z_1|^2|z_2|^2. \end{aligned}$$

In view of this fact we didn't even need to compute the product of our two numbers in the example above.

Theorem 3.3 (Euler's Formula) *For every real number* t,

$$e^{it} = \cos t + i \sin t.$$

Proof First recall that

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}.$$

This means that

$$\begin{aligned} e^{it} &= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(it)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(it)^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} \\ &= \cos(t) + i \sin(t). \end{aligned}$$

∎

Example This is the formula responsible for the famous $e^{i\pi} = -1$. By Euler's Formula we have

$$e^{i\pi} = \cos(\pi) + i \sin(\pi) = -1 + i0 = -1.$$

Corollary 3.4 (Unit Complex Numbers) *Every unit complex number takes the form* e^{it} *for some* t.

Proposition 3.5 *Every complex number can be written in the form* $re^{i\theta}$ *for some* θ.

Proof Let $z = |z| \left(\frac{z}{|z|} \right)$, here $|z| = r$ and $\frac{z}{|z|}$ is a unit complex number so it have the form $e^{i\theta}$ for some θ. ∎

To actually convert from the polar expression to the cartesian (and conversely) some relations. If $z = x + iy$ then

$$\begin{aligned} r^2 &= x^2 + y^2 \\ \theta &= \tan^{-1}(y/x) \end{aligned}$$

Example • The complex number $z = 1 + i$ has a radius of $r = \sqrt{2}$ and an angle of $\theta = \pi/4$. That is $1 + i = \sqrt{2}e^{i\pi/4}$

- The complex number $2e^{\pi/3} = 2(\cos(\pi/3) + i\sin(\pi/3)) = \sqrt{3} + i$.

Now we are at a point where we can justify our claim that every complex number is really an expansion an a rotation.

4 Quaternions

Definition A **quaternion** is a number of the form

$$q = a + bi + cj + dk$$

where a, b, c and d are real numbers and i, j, k[3] The quaternionic numbers i, j and k satisfy

$$i^2 = j^2 = k^2 = ijk = -1. \tag{2}$$

The relations given in equation (2) between i, j and j are just an extremely concise way of specifying special relations in the same way that the relation $i2 = -1$ did for complex numbers.

Here is an example. The multiplicative inverse of i is $-i$. This is because $i(-i) = -i2 = (-1)(-1) = 1$. Similarly, the multiplicative inverses of j and k are $-j$ and $-k$ respectively.

For example from these relations we can prove the following

Proposition 4.1 (Uncompressed Version of Quaternion Relations) *1. The following are how i, j and k relate to each other*

$$\begin{aligned} ij &= k, \\ jk &= i, \\ ki &= j. \end{aligned}$$

2. Unlike the case of complex multiplication, quaternion multiplication is NONCOMMUTATIVE*:*

$$\begin{aligned} ij &= -ji \\ ik &= -ki \\ jk &= -jk \end{aligned}$$

Proof 1.

$$\begin{aligned} ijk = -1 &\implies i^2jk = -i \\ &\implies -jk = -i \\ &\implies jk = i. \end{aligned}$$

Next we show that $ij = k$

$$\begin{aligned} ijk = -1 &\implies ijk2 = -k \\ &\implies -ij = -k \\ &\implies ij = k. \end{aligned}$$

Next we show that $ki = j$. Using the

$$\begin{aligned} ki &= -ki(-1) \\ &= -ki(j2) \\ &= -k(ij)j \\ &= -k^2j \\ &= j. \end{aligned}$$

[3]You can think of the i in the quaternions as the same or different from the i found in the complex numbers it doesn't really matter.

On the first line we introduced two negatives, and the first to second we used that fact that $j2 = -1$, on the second to third we rearranged, on the third to fourth we used the fact that $ij = k$ and the fourth to fifth we used that $ij = k$.

2. Now we use these along with the fact that we know the multiplicative inverses of i,j and k:

$$\begin{aligned} ij = k &\implies iji = ki \\ &\implies iji = j \\ &\implies -iiji = -ij \\ &\implies ji = -ij \end{aligned}$$

the first implication we multiplied by i on the right, the second implication we used the fact that $ki = j$, the third we multiplied both sides by $-i$ and on the fourth we use the fact that $-ii = 1$.

Similarly,

$$\begin{aligned} jk = i &\implies kjk = ik \\ &\implies kjk = j \\ &\implies -kkjk = -kj \\ &\implies jk = -kj. \end{aligned}$$

and,

$$\begin{aligned} ki = j &\implies iki = ij \\ &\implies iki = k \\ &\implies -iiki = -ik \\ &\implies ki = -ik. \end{aligned}$$

Let $q = (1 + 2i + k)$ and $p = (1 + j)$

$$\begin{aligned} qp &= (1 + 2i + k)(1 + j) \\ &= (1 + 2i + k) + (1 + 2i + k)j \\ &= 1 + 2i + k + (j + 2ij + kj) \\ &= 1 + 2i + k + j + 2k - i \\ &= 1 + i + j + 3k. \end{aligned}$$

while

$$\begin{aligned} pq &= (1 + j)(1 + 2i + k) \\ &= (1 + j) + (1 + j)(2i) + (1 + j)k \\ &= (1 + j) + (2i + 2ji) + k + jk \\ &= 1 + j + 2i - 2k + k + i \\ &= 1 + 3i + j - k. \end{aligned}$$

Every quaternion can be written as $q = q_0 + \vec{v}$ where $\vec{v} = xi + yj + zk$ and sometimes these are called the **real part** and the **vector part** of a quaternion number in the same sense that a complex number has a real and imaginary part. It was through studying how these vector parts transform that the operation of the cross product arose.

4.1 Connection to the Usual Vector Operations

We can define the cross product is defined as a function form $\mathbb{R}3 \times \mathbb{R}3 \to \mathbb{R}3$ given by

$$(a_1, b_1, c_1) \times (a_2, b_2, c_2) = (b_2c_2 - c_1b_1, -a_1c_2 + a_2c_1, a_1b_2 - a_2b_1)$$

Identification i,

Proposition 4.2 (Quaternion Multiplication Relates to Dot Products and Cross Products) *Let q*

$$(a_1 i + b_1 j + c_1 k)(a_2 i + b_2 j + c_2 k) = -\vec{v}_1 \cdot \vec{v}_2 + \vec{v}_1 \times \vec{v}_2$$

Proof Expanding the multiplication out on the left hand side above is a little bit of a pain, but it gets things done.

$$\begin{aligned}
(a_1 i + b_1 j + c_1 k)(a_2 i + b_2 j + c_2 k) &= a_1 i(a_2 i + b_2 j + c_2 k) + b_1 j(a_2 i + b_2 j + c_2 k) + c_1 k(a_2 i + b_2 j + c_2 k) \\
&= a_1 a_2 i^2 + a_1 b_2 ij + a_1 c_1 ik + b_1 a_2 ji + b_1 b_2 j^2 + b_1 c_1 jk + c_1 a_2 ki + c_1 b_2 kj + c_1 c_2 k^2 \\
&= a_1 a_2(-1) + a_1 b_2 k + a_1 c_1(-j) + b_1 a_2(-k) + b_1 b_2(-1) \\
&\quad + b_1 c_1(i) + c_1 a_2(j) + c_1 b_2(-i) + c_1 c_2(-1) \\
&= -(a_1 a_2 + b_1 b_2 + c_1 c_2) + (b_2 c_2 - c_1 b_1)i - (a_1 c_2 - a_2 c_1)j + (a_1 b_2 - a_2 b_1)k \\
&= \vec{v}_1 \cdot \vec{v}_2 + \vec{v}_1 \times \vec{v}_2
\end{aligned}$$

$\blacksquare$

Proposition 4.3 $\vec{a} \times b = -\vec{b} \times \vec{a}$

Proof Everything in the computation above will stay the same when flipping $\vec{b}$ and $\vec{a}$ with the exception of the cross terms. These are the terms with any of the following combinations

$$ij, ji, ik, ki, kj, jk.$$

Look at the second line in the computation above (I know a pain but still...) — if we multiplied out $\vec{b}\vec{a}$ rather than $\vec{a}\vec{b}$ the terms connected to ij would be replaced with ji and the terms with ik with be replace with ki etc.

Since these are the only terms that contribute to the cross product, and since $ji = -ij$, $ik = -ki$ etc, we will just pick up an overall minus sign for all of these terms. $\blacksquare$

Proposition 4.4 (Quaternion Formula For the Cross Product)

$$\vec{a} \times \vec{b} = \frac{\vec{a}\vec{b} - \vec{b}\vec{a}}{2}$$

Proof The proof is a straightforward computation,

$$\begin{aligned}
\vec{a} \times \vec{b} &= \frac{1}{2}(-\vec{a} \cdot \vec{b} + \vec{a} \times \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \times \vec{a}) \\
&= \frac{1}{2}(\vec{a} \times \vec{b} - \vec{b} \times \vec{a}) \\
&= \frac{1}{2}(\vec{a} \times \vec{b} + \vec{a} \times \vec{a}).
\end{aligned}$$

In the computations we have just used the fact that $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ and $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.

Example From this we can easily see now that $\vec{a} \times \vec{a} = 0$. This is because

$$\vec{a} \times \vec{a} = \frac{\vec{a}\vec{a} - \vec{a}\vec{a}}{2}.$$

We can also prove that the cross product distributes over sums and that you can pull constants out:

Proposition 4.5 (Distributive Properties) *For and vector $\vec{v}$, $\vec{w}$ and $\vec{u}$, and any constant c we have*

1. $(c\vec{w} + \vec{u}) \times \vec{v} = c\vec{w} \times \vec{v} + \vec{u} \times \vec{v}$.
2. $\vec{v} \times (c\vec{w} + \vec{u}) = c\vec{v} \times \vec{w} + \vec{v} \times \vec{u}$.

Proof 1. The proof is a straight forward computation:

$$\begin{aligned}\vec{v}\times(c\vec{w}+\vec{u}) &= \frac{1}{2}(\vec{v}(c\vec{w}+\vec{u})-(c\vec{w}+\vec{u})\vec{v})\\ &= \frac{1}{2}(c\vec{v}\vec{w}+\vec{v}\vec{u}-c\vec{w}\vec{v}-\vec{u}\vec{v})\\ &= c\frac{1}{2}(\vec{v}\vec{w}-\vec{w}\vec{v})+\frac{1}{2}(\vec{v}\vec{u}-\vec{u}\vec{v})\\ &= c\vec{v}\times\vec{w}+\vec{v}\times\vec{u}.\end{aligned}$$

2. Here we just used the antisymmetric property to flip the terms and introduce a minus sign then use part 1

$$\begin{aligned}(c\vec{w}+\vec{u})\times\vec{v} &= -\vec{v}\times(c\vec{w}+\vec{u})\\ &= -c\vec{v}\times\vec{w}-\vec{v}\times\vec{u}\\ &= c\vec{w}\times\vec{v}+\vec{u}\times\vec{v}.\end{aligned}$$

∎

Like the complex numbers we have an operation of similar to complex conjugation called **quaternion conjugation**: It is defined as follows

$$(a+bi+cj+dk)^* = a-bi-cj-dk.$$

Proposition 4.6 (Norm conjugation formula) $qq^* = |q|^2$

Proof Let $q = a+\vec{q}$, where $\vec{q} = ai+bj+ck$,

$$\begin{aligned}qq^* &= (a+\vec{q})(a+\vec{q})*\\ &= a^2+a\vec{q}-a\vec{q}-\vec{q}\vec{q}\\ &= a^2-(-\vec{q}\cdot\vec{q}+\vec{q}\times\vec{q})\\ &= a^2+\vec{q}\cdot\vec{q}\\ &= a^2+b^2+c^2+d^2\\ &= |q|^2.\end{aligned}$$

Unlike complex conjugation, quaternion conjugation is not linear:

Proposition 4.7 *For any quaternion q and p*

$$(qp)^* = p^*q^*.$$

Proof Let $p = p_0+\vec{p}$ and $q = q_0+\vec{q}$, where p_0 and q_0 are the real parts of the quaternion, and $\vec{p}$ and $\vec{q}$ are the vector parts,

$$\begin{aligned}pq &= (p_0+\vec{p})(q_0+\vec{q})\\ &= p_0q_0+\vec{p}q_0+p_0\vec{q}+\vec{p}\vec{q}\\ &= p_0q_0+q_0\vec{p}+p_0\vec{q}+(-\vec{p}\cdot\vec{q}+\vec{p}\times\vec{q})\\ &= (p_0q_0-\vec{p}\cdot\vec{q})+(q_0\vec{p}+p_0\vec{q}+\vec{p}\times\vec{q}),\end{aligned}$$

on the last line we have separated the product into real and vector parts. This allows us to say

$$(pq)^* = (p_0q_0-\vec{p}\cdot\vec{q})-(q_0\vec{p}+p_0\vec{q}+\vec{p}\times\vec{q}).$$

Now a similar computation shows,

$$\begin{aligned}q^*p^* &= (q_0-\vec{q})(p_0-\vec{p})\\ &= p_0q_0-p_0\vec{q}-q_0\vec{p}+\vec{q}\vec{p}\\ &= p_0q_0-p_0\vec{q}-q_0\vec{p}+(-\vec{q}\cdot\vec{p}+\vec{q}\times\vec{p})\\ &= (p_0q_0-\vec{q}\cdot\vec{p})-p_0\vec{q}-q_0\vec{p}-\vec{p}\times\vec{q}\\ &= (pq)^*\end{aligned}$$

∎

Proposition 4.8

$$|qp|^2 = |q|^2|p|^2$$

Another computation,

Proof

$$\begin{aligned} |qp|^2 &= (qp)^*(qp) \\ &= p^*q^*qp \\ &= p^*|q|^2p \\ &= |q|^2p^*p \\ &= |q|^2|p|^2 \end{aligned}$$

$\blacksquare$

Proposition 4.9 $|\vec{w} \times vecv| = |\vec{v}||\vec{w}|\sin\theta$

Proof

$$\begin{aligned} |\vec{v}\vec{w}|^2 &= |-\vec{v}\cdot\vec{w} + \vec{v}\times\vec{w}| && (3) \\ &= |\vec{v}\cdot\vec{w}|^2 + |\vec{v}+\vec{w}|^2 && (4) \\ &= |\vec{v}|^2|\vec{w}|^2(\cos\theta)^2 + |\vec{v}\times\vec{w}|^2 && (5) \end{aligned}$$

Since $|\vec{v}\vec{w}|^2 = |\vec{v}|^2|\vec{w}|^2$ we get

$$|\vec{v}|^2|\vec{w}|^2 - |\vec{v}|^2|\vec{w}|^2(\cos\theta)^2 = |\vec{v}\times\vec{w}|^2$$

which means

$$|\vec{v}\times\vec{w}| = |\vec{v}|^2|\vec{w}|^2(1-(\cos\theta)^2) = |\vec{v}|^2|\vec{w}|^2(\sin\theta)^2$$

which proves the result. $\blacksquare$