Extra Credit — Math 264

September 23, 2009

The problems are worth 5 points a piece.

1. Let $\vec{a}(t)$ and $\vec{b}(t)$ be vector valued functions of one real variable. Prove the product rule for the cross product and dot products

$$\begin{aligned} \frac{d}{dt}\left[\vec{a}(t)\cdot\vec{b}(t)\right] &= \vec{a}'(t)\cdot\vec{b}(t)+\vec{a}(t)\cdot\vec{b}'(t), \\ \frac{d}{dt}\left[\vec{a}(t)\times\vec{b}(t)\right] &= \vec{a}'(t)\times\vec{b}(t)+\vec{a}(t)\times\vec{b}'(t). \end{aligned}$$

Hints: Consider $\vec{a}(t)$ and $\vec{b}(t)$ as quaternion vectors of the form $\vec{r}(t) = x(t)i + y(t)j + z(t)k$, and apply the definition of the derivative

$$\vec{r}'(t) = \lim_{h\to 0}\frac{\vec{r}(t+h)-\vec{r}(t)}{h}$$

derive a product rule for quaternions:

$$\frac{d}{dt}\left[\vec{a}(t)\vec{b}(t)\right] = \vec{a}'(t)\vec{b}(t)+\vec{a}(t)\vec{b}'(t).$$

The prove of this is very similar to a proof of the product rule is the ordinary case.

After proving this use the way the quaternions multiply to deduce the desired product rules (break up into real and vector parts).

2. Suppose that $u(x, y)$ and $v(x, y)$ come from analytic function:

$$f(z) = f(x+iy) = u(x,y)+iv(x,y).$$

Show that the level sets of $u(x, y)$ and $v(x, y)$ meet orthogonally. (Hint: use the Cauchy Riemann equations to related the partial derivatives of $u(x, y)$ to the partial derivatives of $v(x, y)$.)

Note: In the worksheet part of Homework 6 you showed that the level sets of $u(x, y) = x^2 - y^2 + x$ and $v(x, y) = 2xy + y$ are perpendicular. These come from the analytic function $f(z) = z^2 + z$. To see this note that

$$f(z) = z^2+z = (x+iy)^2+(x+iy) = (x^2-y^2+x)+i(2xy+y) = u(x,y)+iv(x,y).$$

3. A *homogeneous polynomial* of degree N is a polynomial

$$f(x,y)\sum_{k=0}^{n}\sum_{j=0}^{m} a_{kj}x^k y^j$$

which satisfies

$$f(\lambda x,\lambda y) = \lambda^N f(x,y).$$

One can show that all the nonzero terms of such a polynomial are of the from $a_{jk}x^k y^j$ with $k+j = N$ (equivalently, if $k + j \neq N$ then $a_{jk} = 0$.)

Homogeneous The polynomial $x^3y + x^2y^2 + y^4$ is homogeneous of degree four.
Not Homogeneous The polynomial $x^2y + xy + y$ is not homogeneous.

Prove that if $f(x, y) =$ is homogeneous of degree N then

$$x\frac{\partial}{\partial x}[f(x, y)] + y\frac{\partial}{\partial y}[f(x, y)] = Nf(x, y)$$