HOMEWORK 11

17.6: 4, 20, 60
17.7: 10, 20

17.6:4 $\quad r(u,v) = 2\sin u\,\hat{i} + 3\cos u\,\hat{j} + v\,\hat{k}$

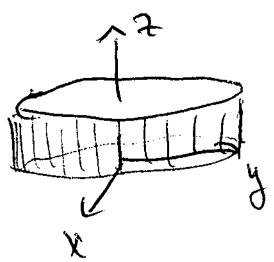
$v \in [0,2]$.

Note: $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \sin(u)^2 + \cos(u)^2 = 1$

So the x & y components satisfy,

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1,$$

Since $z = v$, and runs from 0 to 2, we have that $r(u,v)$ parametrizes part of an elliptical cylinder.



} picture of surface.

17.6:20: Find a parametric representation for the lower half of the ellipsoid:

$$2x^2 + 4y^2 + z^2 = 1.$$

This one you can write as the graph of the function

$$z = -\sqrt{1 - 2x^2 - 4y^2}$$

The parametrization is then

$$r(x,y) = (x, y, f(x,y))$$

$$= (x, y, -\sqrt{1-2x^2-4y^2})$$

where the domain of the parametrization is

$$2x^2 + 4y^2 \leq 1. \,//$$

<u>17.6:60</u> Rotate the circle in the xz-plane w/ center $(b,0,0)$ and radius $a<b$.

$$(b,0,0) + (a\cos(\alpha), 0, a\sin(\alpha))$$

$$= (b + a\cos(\alpha), 0, a\sin(\alpha))$$

" parametrization ~~equation~~ in xz-plane."
$\leftarrow \alpha \in [0, 2\pi)$

To rotate this around the x-axis we let

$$r = (b + a\cos(\alpha))$$

with

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

$\leftarrow \theta \in [0, 2\pi)$

So that

$$x = (b + a\cos(\alpha))\cos(\theta)$$
$$y = (b + a\cos(\alpha))\sin(\theta)$$
$$z = a\sin(\alpha)$$

$\theta \in [0, 2\pi)$
$\alpha \in [0, 2\pi)$

The parametrization is then

$$\vec{r}(\alpha,\theta) = \left((b+a\cos(\alpha))\cos\theta,\ (b+a\cos(\alpha))\sin\theta,\ a\sin(\alpha) \right)_{//}$$

(c)

$$\iint_S dS = \iint_D |\vec{r}_\alpha \times \vec{r}_\theta|\, d\alpha\, d\theta.$$

$$\vec{r}_\alpha = \left(-a\cos(\theta)\sin(\alpha),\ -a\sin(\theta)\overset{\sin}{\cancel{\cos}}(\alpha),\ a\cos(\alpha)\right)$$

$$\vec{r}_\theta = \left(-(b+a\cos(\alpha))\sin(\theta),\ (b+a\cos(\alpha))\cos(\theta),\ 0\right)$$

$$\vec{r}_\alpha \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ -a\cos\theta\sin\alpha & -a\sin\theta\sin\alpha & a\cos\alpha \\ -(b+a\cos(\alpha))\sin\theta & (b+a\cos(\alpha))\cos\theta & 0 \end{vmatrix}$$

$$\begin{aligned} = {} & i\left(-(b+a\cos(\alpha))\cos(\theta)\,a\cos(\alpha)\right) \\ & -j\left(a\cos(\alpha)(b+a\cos(\alpha))\sin\theta\right) \\ & \left.\begin{aligned} +k\Big(& (b+a\cos(\alpha))\cos(\theta)(-a\cos(\theta)\sin(\alpha)) \\ & -(a\sin(\theta)\sin(\alpha))(b+a\cos(\alpha))\sin\theta\Big)\end{aligned}\right] \text{CRAP} \end{aligned}$$

$$\begin{aligned} \Rightarrow |\vec{r}_\alpha \times \vec{r}_\theta|^2 = {} & (b+a\cos(\alpha))^2(\cos\theta)^2(a\cos\alpha)^2 \\ & + (a\cos\alpha)^2(b+a\cos(\alpha))^2(\sin\theta)^2 \\ & + [\quad \text{CRAP} \quad]^2 \end{aligned}$$

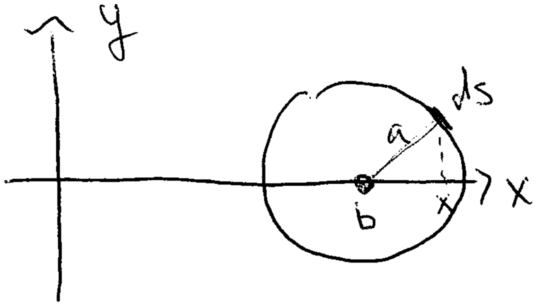
$$
\begin{aligned}
\text{CRAP} = &-ab\cos(\theta)^2\sin(\alpha) - a^2\cos(\alpha)\cos(\theta)^2\sin(\alpha) \\
&-ab\sin(\theta)^2\sin(\alpha) - a^2\cos(\alpha)\sin(\theta)^2\sin(\alpha)
\end{aligned}
$$

$$= -ab\sin(\alpha) - a^2\cos(\alpha)\sin(\alpha)$$

$$
\begin{aligned}
\therefore\ |\vec{r}_\alpha \times \vec{r}_\theta|^2 &= [b + a\cos(\alpha)]^2 a^2\cos(\alpha)^2\left(\cos(\theta)^2 + \sin(\theta)^2\right) \\
&\quad + a^2\sin(\alpha)^2\,[b + a\cos(\alpha)]^2 \\
&= (b + a\cos(\alpha))^2\left[a^2\cos(\alpha)^2 + a^2\sin(\alpha)^2\right] \\
&= a^2(b + a\cos(\alpha))^2.
\end{aligned}
$$

$$
\begin{aligned}
\therefore \iint_S dS &= \iint_D |\vec{r}_\alpha \times \vec{r}_\theta|\,d\alpha\,d\theta \\
&= \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos(\alpha))\,d\alpha\,d\theta \\
&= \int_0^{2\pi}\int_0^{2\pi} ab\,d\alpha\,d\theta + a\int_0^{2\pi}\int_0^{2\pi}\cos(\alpha)\,d\alpha\,d\theta \\
&= 4\pi^2 ab + 2\pi a\left[\sin(\alpha)\Big|_0^{2\pi}\right] \\
&= (2\pi a)(2\pi b).
\end{aligned}
$$

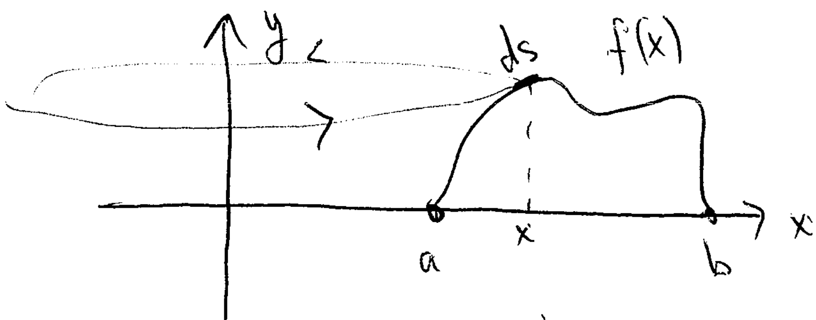
COMPUTATION VIA METHODS IN CALC II



$$(x-b)^2 + y^2 = a^2$$

$$\Rightarrow \quad y = +\sqrt{a^2-(x-b)^2}$$

$$y' = \frac{x-b}{\sqrt{a^2-(x-b)^2}}$$



general surface area formula by surface of revolution method.

$$\int_a^b 2\pi x \, ds = \int_a^b 2\pi x \sqrt{1+f'(x)^2}\, dx$$

$$= \int_{b-a}^{b+a} 2\pi x \sqrt{1+\left(\frac{x-b}{\sqrt{a^2-(x-b)^2}}\right)^2}\, dx$$

$$= \int_{b-a}^{b+a} 2\pi x \sqrt{1+\frac{(x-b)^2}{a^2-(x-b)^2}}\, dx$$

$$= \int_{b-a}^{b+a} 2\pi x a \,/ \sqrt{a^2-(x-b)^2}\, dx$$

$$= 2\pi a \int_{b-a}^{b+a} \frac{x}{\sqrt{a^2-(x-b)^2}}\, dx \quad \cancel{= 2\pi}$$

$$u = x - b.$$

$$= 2\pi a \int_{-a}^{a} \frac{u+b}{\sqrt{a^2-u^2}}\, du$$

$$= 2\pi a \int_{-a}^{a} \frac{u}{\sqrt{a^2-u^2}}\, du + 2\pi a \int_{-a}^{a} \frac{b}{\sqrt{a^2-u^2}}\, du$$

$$= 2\pi a \int_{-a}^{a} \frac{d}{du}\left[\sqrt{a^2-u^2}\right] du + 2\pi \cancel{a} b \int_{-a}^{a} \frac{1}{\sqrt{1-(\frac{u}{a})^2}}\, du$$

$$\sin^{-1}(x) = y \Rightarrow x = \sin(y)$$

$$\Rightarrow 1 = \cos(y)\, y'$$

$$\Rightarrow \frac{1}{\cos(y)} = y'$$

$$\Rightarrow \frac{1}{\sqrt{1-\sin(y)^2}} = y' \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

OR $\frac{d}{du}\left[\sin^{-1}(u)\right] = \frac{1}{\sqrt{1-u^2}}.$ $\Rightarrow \frac{d}{du}\left[\sin^{-1}\left(\frac{u}{a}\right)\right]$

$$= \frac{1}{\sqrt{1-(\frac{u}{a})^2}} \cdot \frac{1}{a}$$

$$= \frac{1}{\sqrt{a^2-u^2}}.$$

$$= 2\pi a \int_{-a}^{a} \frac{d}{du}\left[\sqrt{a^2-u^2}\right] du + 2\pi b \int_{-a}^{a} \frac{d}{du}\left[a \sin^{-1}\left(\frac{u}{a}\right)\right] du$$

$$= 2\pi a \left(\cancel{\sqrt{a^2-u^2}} \Big|_{u=-a}^{u=a} \right)^{\to 0} + 2\pi b \left(a \sin^{-1}\left(\frac{u}{a}\right) \Big|_{-a}^{a} \right)$$

$$= 2\pi b \left(a \sin^{-1}\left(\frac{a}{a}\right) - a \sin^{-1}\left(\frac{-a}{a}\right)\right)$$

$$= 2\pi ba \left(\sin^{-1}(1) - \sin^{-1}(-1) \right)$$

$$= 2\pi ba \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$= 2\pi ba\pi = 2\pi^2 ba,$$

This formula was for the upper half of the circle $\sqrt{a^2-(x-b)^2} = y$, so we need to multiply by two to get

$$\text{SURFACE AREA} = 4\pi^2 ba.$$

17.7: 10

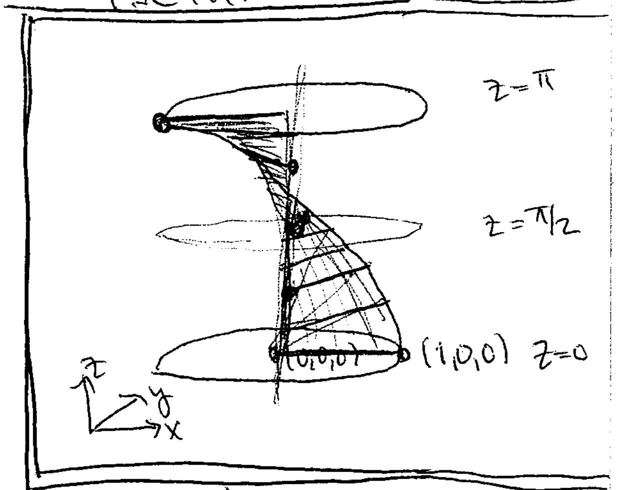
$$\iint_S \sqrt{1+x^2+y^2}\, dS, \qquad S: \ r(u,v) = u\cos(v)\hat{\imath} + u\sin(v)\hat{\jmath} + v\hat{k}$$

$$\underbrace{u \in [0,1],\ v \in [0,\pi]}_{D}.$$

$$\vec{r}_u = \cos(v)\hat{\imath} + \sin(v)\hat{\jmath}$$

$$\vec{r}_v = -u\sin(v)\hat{\imath} + u\cos(v)\hat{\jmath} + \hat{k}.$$

PICTURE OF SURFACE



$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos(v) & \sin(v) & 0 \\ -u\sin(v) & u\cos(v) & 1 \end{vmatrix}$$

$$= \hat{\imath}\left(\cancel{u\cos(v)}\ \sin(v)\right) - \hat{\jmath}\left(\cos(v)\right) + \hat{k}\left(u\cos(v)^2 + u\sin(v)^2\right)$$

$$= \sin(v)\hat{\imath} - \cos(v)\hat{\jmath} + u\hat{k}.$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\sin(v)^2 + \cos(v)^2 + u^2}$$

$$= \sqrt{1+u^2}.$$

$$\iint_S \sqrt{1+x^2+y^2}\, dS = \iint_D \sqrt{1+x(u,v)^2+y(u,v)^2}\ |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

$$= \iint_D \sqrt{1+u^2}\ \sqrt{1+u^2}\, du\, dv$$

$$= \int_0^1 \int_0^{\pi} (1+u^2)\,dv\,du$$

$$= \pi \left(\int_0^1 1+u^2\,du \right)$$

$$= \pi \left(u + \frac{u^3}{3} \Big|_0^1 \right)$$

$$= \frac{4\pi}{3} \,.\,//$$

<u>17.7:20</u>: Compute $\iint \vec{F} \cdot d\vec{S}$ where S is the surface w/ parametrization

From previous problem.

$$\vec{r}(u,v) = u\cos(v)\,\hat{\imath} + u\sin(v)\,\hat{\jmath} + v\,\hat{k}$$

where $u \in [0,1]$, $v \in [0,\pi]$, &

$$\vec{F} = y\,\hat{\imath} + x\,\hat{\jmath} + z^4\,\hat{k}$$

correct outward orientation since k goes upward. (see previous problem)

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\,du\,dv$$

$$= \iint_D (u\sin(v)\,\hat{\imath} + u\cos(v)\,\hat{\jmath} + v^4\,\hat{k}) \cdot \left(\sin(v)\,\hat{\imath} - \cos(v)\,\hat{\jmath} + u\,\hat{k} \right) du\,dv$$

$$= \iint_D (u\sin(v)^2 - u\cos(v)^2 + uv^4)\,du\,dv$$

$$= \int_0^{\pi}\int_0^1 (2u\sin(v)^2 - u + uv^4)\,du\,dv$$

$$= \int_0^{\pi}\int_0^1 u(2\sin(v)^2 - 1 + v^4)\,du\,dv$$

$$= \frac{1}{2}\int_0^{\pi} (2\sin(v)^2 - 1 + v^4)\,dv$$

<u>NOTE:</u>

$$\int_0^{\pi} \sin(v)^2 = \frac{\pi}{2}$$

$$= \frac{1}{2}\left[\pi - \pi + \frac{\pi^5}{5}\right] = \frac{\pi^5}{10}. //$$