Practice Test 3 — Math 264 — Fall 2009

December 2, 2009

1. Find the volume under the surface $x^2y = z$ lying above the domain D in the xy-plane bounded by the curves $x = y^2$ and $y = x^2$.

2. Evaluate the integral
$$\int_0^1 \int_x^1 \sin(y^2) dy dx.$$
Hint: Consider the domain of integration and rewrite the integral so that you integral x first.

3. Compute
$$\int_C \vec{F} \cdot \vec{T} ds$$
where $\vec{F} = x^2\hat{\imath} + y\hat{\jmath} + 2\hat{k}$ and C is the line segment between the points $(1, 0, 1)$ and $(2, 0, -2)$.

4. Determine if the vector field $\vec{F} = xyz\hat{\imath} + xyz\hat{\jmath} + \hat{k}$ is conservative.

5. Compute that potential of the vector field and use it to compute the integral
$$\int_C \vec{F} \cdot \vec{T} ds,$$
where $F = (e^x y + z)\hat{\imath} + e^x\hat{\jmath} + x\hat{k}$

6. Compute the mass of the 3D region under above the xy-plane and below the paraboloid $z = 1 - x^2 - y^2$ which has a density function $\rho(x, y, z) = \rho_0 x^2 + y^2$ (Hint: Convert to Spherical or Cylindrical coordinates)

7. Find the surface area of the segment of the paraboloid $x = y^2 + z^2$ bounded by the plane $x = 2$.

8. Apply Green's Theorem to evaluate the line integral
$$\int_C x^2 y dx - y^2(x + 1)dy$$
where C is the circle of radius 3 in the xy-plane oriented counterclockwise.

9. (a) For any vector field $F = F(x, y) = (A(x, y), B(x, y))$ and any function $u = u(x, y)$ prove
$$\nabla \cdot (fF) = \nabla f \cdot F + f(\nabla \dot{F}).$$

   (b) Let D be two dimensional region with boundary C,
$$\iint_D u\nabla^2 v dV = \int_C (u\nabla v) \cdot N ds - \iint_D \nabla u \cdot \nabla v dV$$
Hint: expand $\iint_D \nabla \cdot (u\nabla v)dV$ in two different ways. (1) Using the second form of Green's Theorem (2) Using the product rule above.

10. (a) Let C_1 be the contour parametrized by $r_1(t) = (\cos(t), \sin(t))$ for $t \in [0, \pi]$ and compute

$$\int_{C_1} \frac{x}{x^2 + y^2} dy + \frac{-y}{x^2 + y^2} dx$$

(b) Let C_2 be the contour parametrized by $r_2(t) = (\cos(-t), \sin(-t))$ for $t \in [0, \pi]$ and compute

$$\int_{C_2} \frac{x}{x^2 + y^2} dy + \frac{-y}{x^2 + y^2} dx$$

11. Compute

$$\int_{-\infty}^{\infty} e^{-x^2} dx.$$

Hint: see 16.4:36.

12. Find the volume of the three dimensional region inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$.

13. Spherical Coordinates are given by

$$\begin{cases} x = \rho \sin(\phi) \cos(\theta) \\ y = \rho \sin(\phi) \sin(\theta) \\ z = \rho \cos(\phi) \end{cases}$$

Using the Jacobian derive the formula derive the formula $dV = \rho^2 \sin(\phi) d\rho d\theta d\phi$

14. Prove that if $\vec{F}(x, y, z) = \nabla u$ where $u = u(x, y, z)$ then for any curve oriented curve C which starts at (x_0, y_0, z_0) and ends at (x_1, y_1, z_1) we have

$$\int_C \vec{F}(x, y, z) \cdot \vec{T} ds = u(x_1, y_1, z_1) - u(x_0, y_0, z_0).$$

Hint: Let $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ be a parametrization of C with $t \in [a, b]$ then proceed evaluating the left hand side. You will end up using $\nabla u(\vec{r}(t) \cdot \vec{r}'(t) = \frac{d}{dt}[u(\vec{r}(t))]$ and applying the fundamental theorem of calculus.

15. Compute the integral $\int_0^1 \int_0^5 x^2 y + e^y dx dy$.