HOMEWORK 6

15.3: 20, 50
15.4: 2, 6
15.5: 8, 53 ← not graded.

15.3: 20 Find the first partial derivatives of the function $z = \tan(xy)$

$$\frac{\partial z}{\partial x} = \sec^2(xy)\frac{\partial}{\partial x}(xy)$$
$$= y\sec^2(xy)$$

$$\frac{\partial z}{\partial y} = \sec^2(xy)\frac{\partial}{\partial y}(xy)$$
$$= \sec^2(xy)x.$$

15.3: 50: Find the first partial derivatives

a) $z = f(x)g(y)$

$$\frac{\partial z}{\partial x} = f'(x)g(y)$$
$$\frac{\partial z}{\partial y} = f(x)g'(y)$$

b) $z = f(xy)$,

$$\frac{\partial z}{\partial x} = f'(xy)\frac{\partial}{\partial x}(xy) = f'(xy)y.$$
$$\frac{\partial z}{\partial y} = f'(xy)x.$$

c) $z = f(x/y)$. $\quad \frac{\partial z}{\partial x} = f'(x/y)\frac{\partial}{\partial x}(x/y)$

$$= f'(x/y)\cdot(1/y)$$

$$\frac{\partial z}{\partial y} = f'(x/y)\frac{\partial}{\partial y}\left(\frac{x}{y}\right)$$

$$= f'(x/y)\,\frac{-x}{y^2}.$$

15.4: 2, Find the equation of the tangent plane for $z = 3(x-1)^2 + 2(y+3)^2 + 7$ at the point $(2,-2,12)$.

LOCAL LINEARIZATION FORMULA:

$$z = f(x_0,y_0) + \nabla f(x_0,y_0)\cdot(x-x_0, y-y_0)$$

(not the same z as above)

where $f(x,y) = 3(x-1)^2 + 2(y+3)^2 + 7$, and $(x_0, y_0, f(x_0,y_0)) = (2,-2,12)$.

$$\frac{\partial f}{\partial x} = 6(x-1) \Rightarrow \frac{\partial f}{\partial x}(2,-2) = 6$$

$$\frac{\partial f}{\partial y} = 4(y+3), \Rightarrow \frac{\partial f}{\partial y}(2,-2) = 4$$

$$\therefore\ z = 12 + (6,4)\cdot(x-2, y+2)$$

is the formula for the tangent plane.

15.4: 6 same setup as previous problem

$$f(x,y) = e^{x^2-y^2}, \quad \text{point} = (1,-1,1).$$

$$\nabla f = \left(2x e^{x^2-y^2}, -2y e^{x^2-y^2}\right)$$

$$\Rightarrow \nabla f(1,-1) = (2,2).$$

$$\begin{aligned} z &= f(1,-1) + \nabla f(1,-1)\cdot(x-1, y+1) \\ &= 1 + (2,2)\cdot(x-1,y+1) \\ &= \underbrace{1 + 2(x-1) + 2(y+1)}_{\text{formula for tangent plane.}} \end{aligned}$$

15.5: 8 Find $\frac{\partial z}{\partial s}$ & $\frac{\partial z}{\partial t}$ for $z = \sin^{-1}(x-y)$ where $x = s^2+t^2$ & $y = 1-2st$.

<u>Recall:</u> $$\frac{d}{dt}\sin^{-1}(t) = \frac{1}{\sqrt{1-t^2}}$$

pf.

$$\sin^{-1}(t) = s \Rightarrow t = \sin(s) \Rightarrow 1 = \cos(s)\frac{ds}{dt}$$

$$\Rightarrow \frac{ds}{dt} = \frac{1}{\cos(s)} = \frac{1}{\sqrt{1-\sin(s)^2}} = \frac{1}{\sqrt{1-t^2}}. //$$

$$\left\{\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}. \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \end{aligned}\right.$$

chain rule formulas

$$\frac{\partial z}{\partial x} = \left(1-(x-y)^2\right)^{-1/2}$$

$$\frac{\partial z}{\partial y} = -\left(1-(x-y)^2\right)^{-1/2}$$

$$\frac{\partial x}{\partial s} = 2s \ , \ \frac{\partial x}{\partial t} = 2t$$

$$\frac{\partial y}{\partial s} = -2t \ , \ \frac{\partial y}{\partial t} = -2s$$

$$x(s,t) - y(s,t) = s^2+t^2 - (1-2st)$$
$$= (s+t)^2 + 1,$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= \left(1-(x-y)^2\right)^{-1/2}(2s) + \left(1-(x-y)^2\right)^{-1/2}(-2t)$$

$$= \frac{2s+2t}{\left(1-(x-y)^2\right)^{1/2}}$$

$$= \frac{2s+2t}{\left(1-\left((s+t)^2-1\right)^2\right)^{1/2}}$$

$$\frac{\partial z}{\partial t} = \left(1-(x-y)^2\right)^{-1/2}(2t) - \left(1-(x-y)^2\right)^{-1/2}(-2s)$$

$$= \frac{2(s+t)}{\left(1-(x-y)^2\right)^{1/2}} = \frac{2(s+t)}{\left(1-\left((s+t)^2+1\right)\right)^{1/2}}$$

$$z_{xx} + z_{yy} = z_{rr} + \frac{1}{r^2} z_{\theta\theta} + \frac{1}{r} z_r$$

15.5:53: $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial z}{\partial r}\frac{\partial r}{\partial x}$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(z_\theta \theta_x + z_r r_x\right)$$

$$= \frac{\partial}{\partial x}(z_\theta \theta_x) + \frac{\partial}{\partial x}(z_r r_x)$$

$$= \frac{\partial}{\partial x}(z_\theta)\theta_x + z_\theta \frac{\partial}{\partial x}(\theta_x)$$

$$+ \frac{\partial}{\partial x}(z_r) r_x + z_r \frac{\partial}{\partial x}(r_x)$$

$$= \left(\frac{\partial z_\theta}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial z_\theta}{\partial r}\frac{\partial r}{\partial x}\right)\theta_x + z_\theta \theta_{xx}$$

$$+ \left(\frac{\partial z_r}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial z_r}{\partial r}\frac{\partial r}{\partial x}\right) r_x + z_r r_{xx}$$

$$= \left(z_{\theta\theta}\theta_x + z_{\theta r} r_x\right)\theta_x + z_\theta \theta_{xx}$$

$$+ \left(z_{r\theta}\theta_x + z_{rr} r_x\right) r_x + z_r r_{xx}.$$

$$\therefore z_{xx} = z_{\theta\theta}\theta_x^2 + 2 z_{\theta r}\theta_x r_x + z_{rr} r_x^2$$

$$+ z_\theta \theta_{xx} + z_r r_{xx}.$$

similarly,

$$z_{yy} = z_{\theta\theta}\theta_y^2 + 2 z_{\theta r}\theta_y r_y + z_{rr} r_y^2$$

$$+ z_\theta \theta_{yy} + z_r r_{yy}.$$

$$\begin{cases} \theta = \tan^{-1}(y/x) \\ r = \sqrt{x^2+y^2} \end{cases} \qquad \begin{cases} x = r\cos\theta \\ y = r\cos\theta \end{cases}$$

COMPUTATION OF PARTIAL DERIVATIVES OF θ

$$\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\left(\tan^{-1}(y/x)\right)$$

$$= \frac{1}{1+(y/x)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

rect — polar

$$= \frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right) = \boxed{\frac{-y}{x^2+y^2}} = \frac{-r\sin\theta}{r^2} = \boxed{\frac{-\sin\theta}{r}}.$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) = \frac{+y}{(x^2+y^2)^2}\frac{\partial}{\partial x}(x^2+y^2)$$

$$= \boxed{\frac{2xy}{(x^2+y^2)^2}} = \frac{2r\cos\theta\, r\sin\theta}{r^4}$$

$$= \boxed{\frac{2\cos\theta\sin\theta}{r^2}}.$$

$$\frac{\partial \theta}{\partial y} = \frac{\partial}{\partial y}\left(\tan^{-1}(y/x)\right)$$

$$= \frac{1}{1+(y/x)^2}\frac{\partial}{\partial y}\left(y/x\right)$$

$$= \frac{1}{1+(y/x)^2}\left(1/x\right) = \boxed{\frac{x}{x^2+y^2}} = \frac{r\cos\theta}{r^2} = \boxed{\frac{\cos\theta}{r}}$$

$$\frac{\partial^2 \theta}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2}$$ (similar to the computation of θ_{xx})

$$= \boxed{\frac{-2\cos\theta\sin\theta}{r^2}}.$$

COMPUTATION OF PARTIAL DERIVATIVES OF r

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2}\right)$$

$$= \frac{1}{2}(x^2+y^2)^{-1/2}\frac{\partial}{\partial x}(x^2+y^2)$$

$$= \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \boxed{\cos\theta}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$$

$$= \left(\frac{(1)\sqrt{x^2+y^2} - \frac{x}{\sqrt{x^2+y^2}}(x)}{x^2+y^2}\right)\left(\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{x^2+y^2-x^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{r^2(\sin\theta)^2}{r^3} = \boxed{\frac{(\sin\theta)^2}{r}}$$

similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \boxed{\sin\theta}, \qquad \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{(x^2+y^2)^{3/2}} = \boxed{\frac{(\cos\theta)^2}{r}},$$

~~Now lets convert all of these expressions into polar coordinates,~~

$$\Rightarrow z_{xx} + z_{yy} = \left(z_{\theta\theta}\theta_x^2 + 2z_{\theta r}\theta_x r_x + z_{rr} r_x^2 + z_\theta \theta_{xx} + z_r r_{xx} \right) + \left(z_{\theta\theta}\theta_y^2 + 2z_{\theta r}\theta_y r_y + z_{rr} r_y^2 + z_\theta \theta_{yy} + z_r r_{yy} \right)$$

$$= z_{\theta\theta}\left(\theta_x^2 + \theta_y^2\right) + 2z_{\theta r}\left(\theta_x r_x + \theta_y r_y\right) + z_{rr}\left(r_x^2 + r_y^2\right) + z_\theta\left(\theta_{xx} + \theta_{yy}\right) + z_r\left(r_{xx} + r_{yy}\right)$$

$$= z_{\theta\theta}\left(\left(\frac{-\sin\theta}{r}\right)^2 + \left(\frac{\cos\theta}{r}\right)^2 \right) + 2z_{\theta r}\left(\left(-\frac{\sin\theta}{r}\right)(\cos\theta) + \left(\frac{\cos\theta}{r}\right)(\sin\theta) \right) + z_{rr}\left((\cos\theta)^2 + (\sin\theta)^2 \right) + z_\theta\left(\left(\frac{2\cos\theta\sin\theta}{r^2}\right) + \left(-\frac{2\cos\theta\sin\theta}{r^2}\right) \right) + z_r\left(\left(\frac{(\sin\theta)^2}{r}\right) + \left(\frac{(\cos\theta)^2}{r}\right) \right)$$

$$= \frac{z_{\theta\theta}}{r^2} + z_{rr} + \frac{z_r}{r}. \quad //$$