HW 07

§15.6: 8, 22, 49

15.6:8: Find ∇f, eval $\nabla f(x_0, y_0)$, find $(D_{\vec{u}} f)(x_0, y_0)$.

$$f(x,y) = y^2/x$$
$$(x_0, y_0) = (1,2)$$
$$\vec{u} = \frac{1}{3}(2i + \sqrt{5}\,j)$$

soln

$$\nabla f = (-y^2/x^2, 2y/x)$$

$$\nabla f(1,2) = (-4,4)$$

$$(D_{\vec{u}} f)(1,2) = \nabla f(1,2) \cdot \vec{u}$$
$$= (-4i + 4j) \cdot \left(\frac{1}{3}(2i + \sqrt{5}\,j)\right)$$
$$= \frac{1}{3}(-8 + 4\sqrt{5}).$$

15.6:22: Find the maximal rate of change of $f(p,q) = qe^{-p} + pe^{-q}$ at the point $(0,0)$ & the direction in which it occurs.

soln: the max rate of change at (x_0, y_0) is always $|\nabla f(x_0, y_0)|$ and the direction it occurs is always $\nabla f(x_0, y_0) / |\nabla f(x_0, y_0)|$.

$$\nabla f = \left(-q e^{-p} + e^{-q},\ e^{-p} - p e^{-q}\right),$$

$$\nabla f(0,0) = (1,1) \Rightarrow \begin{pmatrix}\text{Max Rate}\\ \text{of Change}\end{pmatrix} = \sqrt{2}$$

$$\begin{pmatrix}\text{Direction}\\ \text{of Max}\\ \text{Rate}\end{pmatrix} = \frac{(1,1)}{\sqrt{2}}.$$

<u>15.6 #49</u>: Prove that the plane tangent to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at the point (x_0, y_0, z_0) can be written as

$$\frac{x_0}{a^2} x + \frac{y_0}{b^2} y + \frac{z_0}{c^2} z = 1,$$

<u>proof</u>: This is a special case of a tangent plane to a level surface $g(x,y,z) = c$. The formula for these is

$$\nabla g(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

Here $g(x,y,z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}$.

$$\Rightarrow \nabla g = \frac{2x}{a^2} i + \frac{2y}{b^2} j + \frac{2z}{c^2} k$$

$$\Rightarrow \nabla g(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0)$$

$$= \left(\frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{2z_0}{c^2}\right) \cdot (x - x_0, y - y_0, z - z_0)$$

$$= \frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) + \frac{2z_0}{c^2}(z - z_0)$$

$$= 2\left(\frac{x_0}{a^2}x + \frac{y_0}{b^2}y + \frac{z_0}{c^2}z - \left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2}\right)\right)$$

$$= 2\left(\frac{x_0}{a^2}x + \frac{y_0}{b^2}y + \frac{z_0}{c^2}z - 1\right).$$

use the fact that (x_0, y_0, z_0) is on the ellipsoid.

$$\Rightarrow 2\left(\frac{x_0}{a^2}x + \frac{y_0}{b^2}y + \frac{z_0}{c^2}z - 1\right) = 0$$

$$\Rightarrow \frac{x_0}{a^2}x + \frac{y_0}{b^2}y + \frac{z_0}{c^2}z - 1 = 0$$

$$\Rightarrow \boxed{\frac{x_0}{a^2}x + \frac{y_0}{b^2}y + \frac{z_0}{c^2}z = 1}.$$