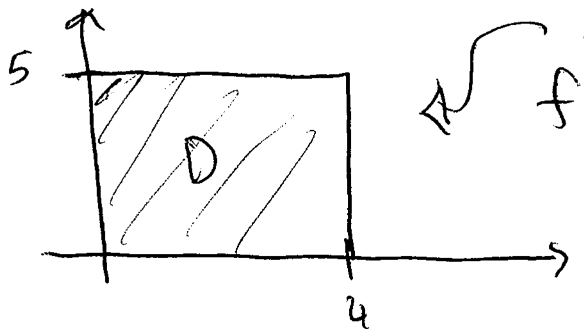
HW 08 $\qquad$ 15.7: 32, 34
15.8: 4, 16, 32

<u>15.7:32</u>: Find the absolute minimum & maximum values of $f(x,y) = 4x+6y-x^2-y^2$ on the set $D = \{(x,y) \mid x \in [0,4] \;\&\; y \in [0,5]\}$

<u>Soln</u>:



the set D is compact so f ~~it~~ will achieve its min and max values. We need to check the interior and the boundary.

$$f_x = 4-2x = 0$$
$$f_y = 6-2y = 0$$

$\Rightarrow$ $x_0 = 2$ & $y_0 = 3$ is the only critical point in the interior. (Note that $13 = f(2,3)$ is going to be the absolute maximum since the graph of f is an upside down paraboloid)

We need to check the boundary:

$$\frac{\partial}{\partial x} f(x,0) = 4-2x, \qquad \frac{\partial}{\partial y} f(0,y) = 6-2y,$$
$$\frac{\partial}{\partial x} f(x,5) = 4-2x, \qquad \frac{\partial}{\partial y} f(4,y) = 6-2y,$$

$\Rightarrow$ $(2,0), (2,5), (0,3), (4,3)$ are critical points.

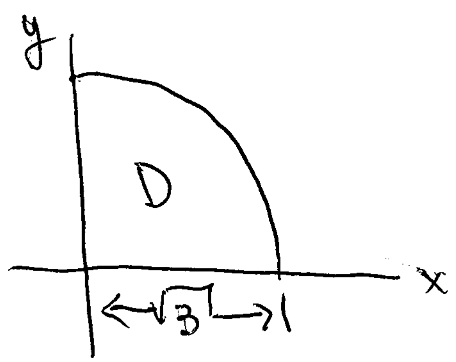
We also need to check the corners $(0,0)$, $(4,0)$, $(4,5)$ & $(0,5)$, so

$$f(2,0) = 4 + 0 = 4$$
$$f(2,5) = 4 + 5 = 9$$
$$f(0,3) = 0 + 9 = 9$$
$$f(4,3) = 0 + 9 = 9$$
$$f(0,0) = 0 + 0 = 0$$
$$f(4,0) = 0 + 0 = 0$$
$$f(4,5) = 0 + 5 = 5$$
$$f(0,5) = 0 + 5 = 5$$

$\Rightarrow f(4,0) = f(0,0) = 0$ are the absolute minima.

15.7:34 same as above only with $f(x,y) = xy^2$ and $D = \{(x,y) \mid x \geq 0 \text{ \& } y \geq 0 \text{ \& } x^2+y^2 \leq 3\}$.

soln.



the function is ~~strictly~~ non-negative, since $f(0,0)=0$ ~~so~~ we have that zero is the absolute minimum of f on the domain.

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow$$ $(x_0, y_0) = (0,0)$ is the only critical point, ~~here at~~ so we don't need to worry about the interior!

cont →

parametrize the outer boundary of D:

$$\vec{r}(t) = (\sqrt{3}\cos t, \sqrt{3}\sin t), \quad t \in [0, \tfrac{\pi}{2}].$$

$$\frac{d}{dt}\left[f(\vec{r}(t))\right] = \frac{d}{dt}\left[3^{3/2}\cos(t)\sin(t)^2\right]$$
$$= 3^{3/2}(-\sin(t)^3 + 2\cos(t)^2\sin(t))$$
$$= 3^{3/2}(-\sin(t)^3 + 2(1-\sin(t)^2)\sin(t))$$
$$= 3^{3/2}(-3\sin(t)^3 + 2\sin(t))$$
$$= 3^{3/2}(\sin(t))(-3\sin(t)^2 + 2)$$

~~or~~ so we have critical points when $\sin(t) = 0$ (~~at~~ a case we don't care about) and when $(-3\sin(t)^2+2) = 0$,

$$-3\sin(t)^2 + 2 = 0 \Rightarrow \sin(t)^2 = \frac{2}{3} \Rightarrow \sin(t_0) = \pm\sqrt{\frac{2}{3}}.$$

since we are only looking at points in the first quadrant we must have $\sin(t_0) = +\sqrt{\frac{2}{3}}$. Thus means $t_0 = \sin^{-1}\left(\sqrt{\frac{2}{3}}\right)$ is a critical point.

~~the~~ since the set is compact we know it must achieve a max and min & this must be the point we are looking for, ~~but~~ lets ~~do the math~~ compute its value:

$$\vec{r}(t_0) = \vec{r}\left(\sin^{-1}\left(\sqrt{\tfrac{2}{3}}\right)\right)$$
$$= \left(\sqrt{3}\cos\left(\sin^{-1}\left(\sqrt{\tfrac{2}{3}}\right)\right), \sqrt{3}\sin\left(\sin^{-1}\left(\sqrt{\tfrac{2}{3}}\right)\right)\right)$$
$$= \left(\sqrt{3}\sqrt{1-\sin\left(\sin^{-1}\left(\sqrt{\tfrac{2}{3}}\right)\right)^2}, \sqrt{3}\left(\sqrt{\tfrac{2}{3}}\right)\right)$$
$$= \left(\sqrt{3}\sqrt{1-2/3}, \sqrt{2}\right)$$
$$= (1, \sqrt{2}).$$

$f(1, \sqrt{2}) = 2$ is the maximum value.

<u>15.8:4</u>: Solve the Lagrange multiplier problem subject to the given constraint

$$\begin{cases} f(x,y) = 4x + 6y \\ x^2 + y^2 = 13 \end{cases}$$

<u>soln</u>. Since the set $\{(x,y) \mid x^2+y^2 = 13\}$ is compact we ~~will~~ will be able to solve for an absolute max & min.

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = c \end{cases} \Rightarrow \begin{cases} 4 = 2\lambda x \\ 6 = 2\lambda y \\ x^2+y^2 = 13. \end{cases} \Rightarrow x = 2/\lambda,\ y = 3/\lambda$$

$$\Rightarrow (2/\lambda)^2 + (3/\lambda)^2 = 13 \Rightarrow \frac{1}{\lambda^2} = \frac{13}{13} = 1 \Rightarrow \lambda = \pm 1.$$

when $\lambda = 1$, we get the solution $(x_0, y_0) = (2, 3)$

when $\lambda = -1$ we get the solution $(x_1, y_1) = (-2, -3)$.

MAX: $f(2,3) = 30$
MIN: $f(-2,3) = -30$.

15.8 #16: Lagrange Multiplier Problem:

$$\begin{cases} f(x,y,z) = 3x - y - 3z \\ x + y - z = 0 \\ x^2 + 2z^2 = 1 \end{cases}$$

soln: The intersection of the plane $x+y-z=0$ and the elliptical cylinder $x^2+2z^2=1$ is a compact set so ~~max~~ f will achieve its max & min.

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g = 0 \\ h = 1 \end{cases} \Rightarrow \begin{cases} 3 = \lambda + 2\mu x & (1) \\ -1 = \lambda + \mu(0) & (2) \\ -3 = -\lambda + 2\mu z & (3) \\ x + y - z = 0 & (4) \\ x^2 + 2z^2 = 1 & (5) \end{cases}$$

$\Rightarrow \lambda = -1$. Equations (3) & (1) give $x = 2/\mu$ and $z = -1/\mu$. Putting these into equation (5) gives $(2/\mu)^2 + 2(-1/\mu)^2 = 1 \Rightarrow \left(\frac{1}{\mu}\right)^2 = 1/6 \Rightarrow \frac{1}{\mu} = \pm\frac{1}{\sqrt{6}}$.

~~when $\mu = +1/\sqrt{6}$: $x =$~~

when $\mu = +\sqrt{6}$: $x = 2/\sqrt{6}$, $z = -1/\sqrt{6}$. Using eqn (4) gives $2/\sqrt{6} + y + 1/\sqrt{6} = 0 \Rightarrow y = -3/\sqrt{6}$. So that we have the critical point $(2/\sqrt{6}, -3/\sqrt{6}, -1/\sqrt{6})$.

<u>when $\mu = -\sqrt{6}$</u>: $x = -2/\sqrt{6}$, $y = 1/\sqrt{6}$. using eqn (4) gives $-2/\sqrt{6} + y - 1/\sqrt{6} = 0 \Rightarrow y = 3/\sqrt{6}$. So we get the critical point $(-2/\sqrt{6}, 3/\sqrt{6}, 1/\sqrt{6})$.

$$f(2/\sqrt{6}, -3/\sqrt{6}, -1/\sqrt{6}) = 3(2/\sqrt{6}) - (-3/\sqrt{6}) - 3(-1/\sqrt{6})$$
$$= \frac{6+3+3}{\sqrt{6}}$$
$$= 2\sqrt{6} \qquad \underline{\text{MAX}}$$

$$f(-2/\sqrt{6}, +3/\sqrt{6}, 1/\sqrt{6}) = -2\sqrt{6} \qquad \underline{\text{MIN}}$$

<u>15.8:32</u>: (15.7: 44) Find three positive numbers whose sum is 12 & the sum of whose squares is as small as possible.

<u>soln</u>:
$$x + y + z = 12$$
$$f(x,y,z) = x^2 + y^2 + z^2$$

Note that constraint is not a compact set so we are not guaranteed a max & min. ~~The critical point we will ex~~

If you think about it, [illegible] that this is [illegible] you can see the minimum

$$\begin{cases} Df = \lambda Dg, \\ g = 12, \end{cases} \Rightarrow \begin{cases} 2x = \lambda \\ 2y = \lambda \\ 2z = \lambda \\ x + y + z = 12 \end{cases} \Rightarrow x = y = z = \lambda/2$$

$$\Rightarrow 3(\lambda/2) = 12 \Rightarrow \lambda = 8 \Rightarrow x = y = z = 4.$$