HOMEWORK 11

§16.8: 22, 30
§16.9: 14

16.8:22 $\quad H = \{ (x,y,z) \mid x^2+y^2+z^2 \le 9 \ \& \ z \ge 0 \}$

$$\iiint_H (9-x^2-y^2)\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{3} (9-\rho^2\sin(\phi)^2)\,\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$

$$= 2\pi \int_0^{\pi/2}\int_0^{3} 9\rho^2\sin(\phi) - \rho^4\sin(\phi)^3\,d\rho\,d\phi$$

$$= 2\pi\left[9\left[\int_0^{\pi/2}\sin\phi\,d\phi\right]\left[\int_0^3 \rho^2\,d\rho\right] - \left[\int_0^{\pi/2}\sin(\phi)^3\,d\phi\right]\left[\int_0^3 \rho^4\,d\rho\right]\right]$$

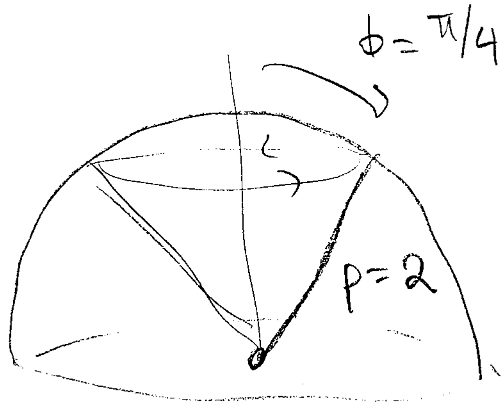
$$= 2\pi\left[9\left(\frac{\pi}{2}\right)(3) - \left(\frac{\pi}{2}\cdot\frac{2}{3}\right)\frac{3^5}{5}\right]$$

$$= \pi^2\left(27 - \frac{81}{5}\right).$$

using

$$\int_0^{\pi/2} \sin(\theta)^n\,d\theta = \begin{cases} \left(\frac{\pi}{2}\right)\dfrac{1\cdot 3\cdot \cdots \cdot(n-1)}{2\cdot 4\cdot \cdots \cdot n}, & n \text{ even} \\ \frac{\pi}{2}\,\dfrac{2\cdot 4\cdot \cdots \cdot n-1}{3\cdot 5\cdot 7\cdots n}, & n \text{ odd.} \end{cases}$$

16.8 : 30 Find the volume of the solid that lies within the sphere $x^2+y^2+z^2=4$, above the xy-plane & above the cone $z=\sqrt{x^2+y^2}$.



$$\left.\begin{array}{l}\rho\in[0,2]\\ \phi\in[0,\pi/4]\\ \theta\in[0,2\pi]\end{array}\right\}$$ problem has constant bounds of integration in spherical coordinates.

$$\mathrm{vol}(E)=\iiint_E dV=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2}\rho^2\sin(\phi)\,d\rho\, d\phi\, d\theta$$

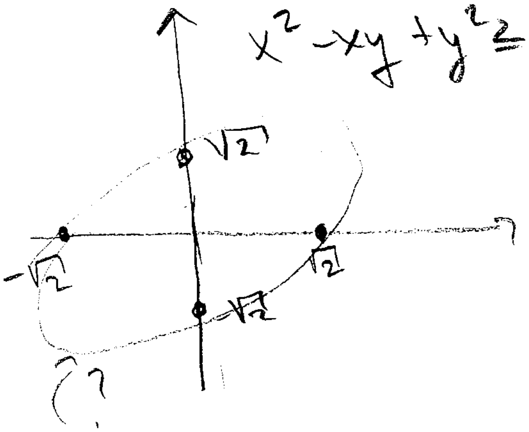
$$=\left(\int_0^{2\pi}d\theta\right)\left(\int_0^{\pi/4}\sin(\phi)\,d\phi\right)\left(\int_0^{2}\rho^2\,d\rho\right)$$

$$=2\pi\left(-\cos(\pi/4)-(-\cos(0))\right)\left(\frac{8}{3}\right)$$

$$=\frac{16\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right).$$ //

16.9.14

$x^2 - xy + y^2 \leq 2$



$$\iint_R (x^2 - xy + y^2)\, dA$$

$$x = \sqrt{2}\,u - \sqrt{\tfrac{2}{3}}\,v$$
$$y = \sqrt{2}\,u + \sqrt{\tfrac{2}{3}}\,v$$

$$\begin{aligned}
x^2 - xy + y^2 &= \left(\sqrt{2}\,u - \sqrt{\tfrac{2}{3}}\,v\right)^2 \\
&\quad - \left(\sqrt{2}\,u - \sqrt{\tfrac{2}{3}}\,v\right)\left(\sqrt{2}\,u + \sqrt{\tfrac{2}{3}}\,v\right) \\
&\quad + \left(\sqrt{2}\,u + \sqrt{\tfrac{2}{3}}\,v\right)^2 \\
&= 2u^2 - \cancel{2(\sqrt{2}\,u)(\sqrt{\tfrac{2}{3}}\,v)} + \tfrac{2}{3}v^2 \\
&\quad - \left(2u^2 - \tfrac{2}{3}v^2\right) \\
&\quad + 2u^2 + \cancel{2(\sqrt{2}\,u)(\sqrt{2/3})}\,v + \tfrac{2}{3}v^2 \\
&= 2u^2 + \tfrac{6}{3}v^2 = 2u^2 + 2v^2
\end{aligned}$$

So the region $x^2 - xy + y^2 \leq 2$ becomes the region $u^2 + v^2 \leq 1$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$$
$$= \det\begin{bmatrix} \sqrt{2} & -\sqrt{2/3} \\ \sqrt{2} & +\sqrt{2/3} \end{bmatrix} = 2\sqrt{2}\cdot\sqrt{2/3} = 4/\sqrt{3}.$$

$$\iint_R (x^2 - xy + y^2)\,dA = \iint_S (u^2 + v^2)\,\frac{4}{\sqrt{3}}\,du\,dv$$

$$= \frac{4}{\sqrt{3}} \int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta$$

$$= \frac{2\pi}{\sqrt{3}}.$$