TEST 1 — Math 264 — Fall 2010

September 29, 2010

1. (a) What operation can be used to find the angle between two vectors? 4 pts
   (b) Find the angle between $\vec{v} = (1, 1, 0)$ and $\vec{w} = (0, 1, -1)$. 3pts
   (c) Find the angle between $\vec{v} = (3, 3, 0)$ and $\vec{w} = (0, 10, -10)$ (think about this for a second). 3pts

a) either the dot or cross product.

b) $\theta = \cos^{-1}\left(\frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ or 60°

c) It is the same, $\pi/3$. These vectors are just multiples of vectors from the previous problem.

2. Let $\vec{v} = (1, 1, 1)$ and $\vec{w} = (1, 0, 0)$
   (a) Find the vector projection of $\vec{v}$ onto $\vec{w}$. 5pts
   (b) Is $\text{proj}_{\vec{w}}(\vec{v})$ a unit vector? 5pts

a) projecting onto $\hat{i}$ gives you $(1,0,0)$. (This is easy)

b) yes, $(1,0,0)$ is a unit vector since $|(1,0,0)|=1$.

you can also use the formula

$$\text{proj}_{\vec{w}}(\vec{v}) = \left(\vec{v} \cdot \frac{\vec{w}}{|\vec{w}|}\right)\frac{\vec{w}}{|\vec{w}|}$$

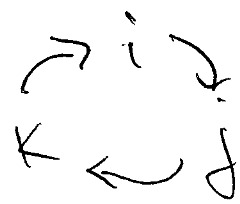
to get this answer.

3. Compute the following cross products

(a) $i \times j$ ~~3pts~~ 3pts

(b) $(k + j) \times i$ ~~3pts~~ 3pts

(c) $(2i + 3k) \times (k - i)$ ~~4pts~~ 4pts



a) $i \times j = k$

b) $(k+j) \times i = k \times i + j \times i = j + (-k) = j - k$

c) $(2i+3k) \times (k-i) = 2i \times k - 2i \times i + 3\, k \times k - 3\, k \times i$

$= -2j - 0 + 0 - 3j$

$= -5j$

You can also do these using the determinant formula.

4. Find the angle between the planes in $\mathbb{R}^3$ defined by $x + y + 2\pi = 0$ and $2y - 2z - 1 = 0$.

The angle between the planes is just the angle between their normal vectors:

$\vec{n}_1 = i + j = (1, 1, 0)$ 5pts

$\vec{n}_2 = +2j - 2k = (0, 2, -2)$

the angle between these will be the same as on problem 1.

$\theta = 60^\circ$ or $\frac{\pi}{3}$. 5pts

5. Find the values of α such that the vectors $(1+\alpha, 2, \alpha^2 - 1)$ and $(\alpha - 1, \alpha, -1)$ are orthogonal. 6 pts

orthogonal = perpendicular.

Two vectors are perpendicular iff their dot product is zero, so we need to find the α such that

$$\underbrace{\begin{aligned} 0 &= (1+\alpha, 2, \alpha^2-1)\cdot(\alpha-1, \alpha, -1) \\ &= (\alpha^2-1) + (2\alpha) - (\alpha^2-1) \\ &= 2\alpha. \end{aligned}}_{2\text{pts}}$$

$\Rightarrow \alpha = 0.$ 2pts

6. Find a parametrization of the line which is the intersection of the planes $x+y+z+1=0$ and $x = z+2$.

Let parametrize using z as our parameter — this means solving for everything in terms of z:

$$\begin{cases} x+y+z-1=0 \\ x = z+2 \end{cases}$$

Some reasonable procedure = 7 pts

$\Rightarrow (z+2)+y+z-1=0 \Rightarrow y = -2z-1$

$$\Rightarrow \begin{cases} x = z+2 \\ y = -2z-1 \\ z = z \end{cases}$$

Answer 3pts

If you would like to relabel your parameter t you have $\vec{r}(t) = (t+2, -2t-1, t)$. You could have also done this by taking cross products of the normals to find a $\vec{v}$, ~~then~~ & using a point on the intersection to get $\vec{r_0}$ & form $\vec{r}(t) = \vec{r_0} + \vec{v}t$.

7. Consider the curve parametrized by $\vec{r}(t) = (t, t^2, 2)$ where $t \in (-\infty, \infty)$. Find a parametrization of the line tangent to the curve at the point $(2, 4, 2)$.

$\vec{r}(t)$ hits that point at $t=2$: $\vec{r}(2) = (2, 2^2, 2)$
+2 $= (2, 4, 2)$.

The derivative of $\vec{r}(t)$ is +5

$\vec{r}\,'(t) = (1, 2t, 0)$.

At $t=2$ we have $\vec{r}\,'(2) = (1, 4, 0)$. The formula for the tangent line (local linear approx) is

$$\vec{l}(t) = \vec{r}(t_0) + \vec{r}\,'(t_0)(t-t_0) = (2,4,2) + (1,4,0)(t-2).$$

+3

8. Find the arclength of the curve parametrized by $\vec{\gamma}(t) = (t^3/3, t^2/\sqrt{2}, t)$ where $t \in [0, 1]$. Hint: $(t^4 + 2t^2 + 1) = (t^2 + 1)^2$.

$$\int_C ds = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt \quad +7 \text{ pts}$$

$$= \int_0^1 \sqrt{(t^2)^2 + (\sqrt{2}\,t)^2 + (1)^2}\, dt \quad +3 \text{ pts}$$

$$= \int_0^1 \sqrt{(t^2+1)^2}\, dt$$

$$= \int_0^1 (t^2+1)\, dt = \frac{t^3}{3} + t \Big|_{t=0}^{t=1} = \frac{1}{3} + 1 = \frac{4}{3}.$$

9. Let $\vec{r}(t) = (a(t), b(t))$ and $\vec{R}(t) = (A(t), B(t))$. Show

$$\frac{d}{dt}[\vec{r}(t) \cdot \vec{R}(t)] = \vec{r}'(t) \cdot \vec{R}(t) + \vec{r}(t) \cdot \vec{R}'(t).$$

$$\frac{d}{dt}\left[\vec{r}(t) \cdot \vec{R}(t)\right] = \frac{d}{dt}\left[a(t)A(t) + b(t)B(t)\right]$$

5pts

$$= \left(a'(t)A(t) + a(t)A'(t)\right) + \left(b'(t)B(t) + B'(t)b(t)\right)$$

$$= \left[a'(t)A(t) + b'(t)B(t)\right] + \left[a(t)A'(t) + B'(t)b(t)\right]$$

5pts

$$= (a'(t), b'(t)) \cdot (A(t), B(t)) + (a(t), b(t)) \cdot (A'(t), B'(t))$$

$$= \vec{r}'(t) \cdot \vec{R}(t) + \vec{r}(t) \cdot \vec{R}'(t).$$

10. Find a parametrization of the line passing through the points $(1, 2, 3)$ and $(-3, -2, -1)$.

Easy Way:

$$\vec{l}(t) = (1-t)P_1 + tP_2$$

$$= (1-t)(1,2,3) + t(-3,-2,-1).$$

(you can expand this out if you want, but you don't need to).

You could also take P_1 as your starting point & $\overrightarrow{P_1P_2}$ as your velocity vector (or some variant of this).

reasonable procedure 7 pts

answer/application of procedure 3 pts