$\boxed{\text{LAST TIME:}}$ optimization

Tuesday 10/4/16

↳ To find critical points, one solves

$\nabla f = 0$

↳ In two variables, (if $f = f(x,y)$) there was the 2nd derivative test for determining if the critical points are local extrema.

2nd Derivative Test:

Suppose $\nabla f(a,b) = 0$ And define

$$D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}(a,b)^2$$

Hessian Determinant

★Remark★ → $f_{xy} = f_{yx}$ for differentiable functions

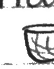
Phase 1:

$D > 0 \Rightarrow (a,b)$ is local extrema

$D < 0 \Rightarrow$ the graph of f at (a,b) has a saddle

$D = 0 \Rightarrow$ test is inconclusive

Phase 2: If $D > 0$ we apply the 2nd derivative test in one variable,

$f_{xx}(a,b) > 0 \Rightarrow$ concave up (local min) 

$f_{xx}(a,b) < 0 \Rightarrow$ concave down (local max) 

★Remark★ → you can use f_{yy} to do this too.

TODAY: Lagrange multipliers & Extreme Value Theorem

Lagrange multipliers

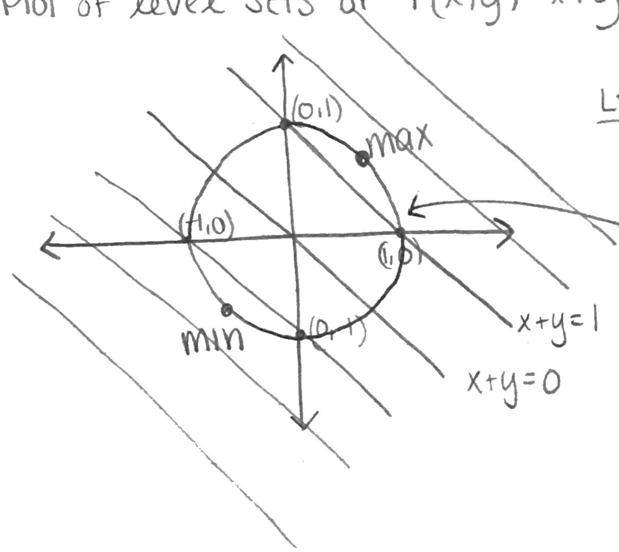
baby example problem: Optimize the function $f(x,y) = x+y$
subject to the constraint, $x^2+y^2=1 \leftarrow g(x,y)$

Terminology:
- optimize $\rightarrow$ Find local max or min
- Constraint $\rightarrow$ Only look at values of the function where $x^2+y^2=1$

This example problem is a "two variable, Lagrange multiplier problem with one constraint."

General Lagrange multiplier problem in two vars w/ one constraint:
optimize a given $f(x,y)$ subject to a given constraint $g(x,y)=k$ where $k \in \mathbb{R}$

Lets look at our "baby example":
Plot of level sets of $f(x,y)=x+y$



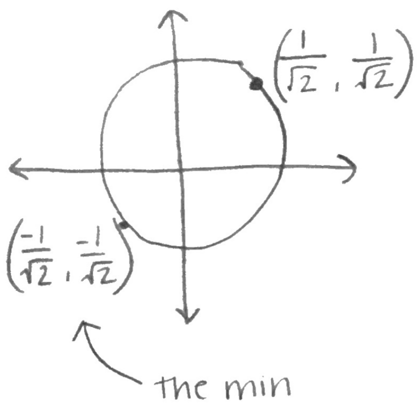
Level Sets:
$x+y=c$ for some $c \in \mathbb{R}$

constraint:
$x^2+y^2=1$

★ Recall
- $\hookrightarrow \nabla f(x,y)$ at a point is the direction of maximal increase
- $\hookrightarrow \nabla g$ is normal to the curve defined by $g(x,y)=k$ or $g(x,y)=0$

$g(x,y)=x^2+y^2-1$.
If $g(x,y)=k$ we can use $G(x,y)=g(x,y)-k$ and look at $G(x,y)=0$ rather than $g(x,y)=0$.

We can see from the picture the max is



The extreme values are where the level sets of $f(x,y)$ are tangent to the constraint $g(x,y) = k$.

In math speak:

$$\underbrace{\nabla f}_{\text{direction of max increase}} = \lambda \overbrace{\nabla g}^{\text{normal vector of curve}}$$

This is saying they are the same up to a constant multiple

This idea works in general.

Solution of 1 constraint Lagrange multiplier Problem:

Solving the system of equations,

$$\left.\begin{aligned} \nabla f &= \lambda \nabla g \\ g(x,y) &= k \end{aligned}\right\}$$

gives candidates for max/min. Plugging them into $f(x,y)$ and taking the max & min solves the problem

Solution to our baby problem:

$$\begin{cases} f(x,y) = x+y & \leftarrow \text{function we want to optimize} \\ x^2+y^2 = 1 & \leftarrow \text{constraint} \end{cases}$$

OR

$g(x,y) = x^2+y^2$	$G(x,y) = x^2+y^2-1$
$k=1$	$k=0$

SET UP OUR SYSTEM:

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 1 \end{cases} \iff \begin{cases} (1,1) = \lambda(2x, 2y) \\ x^2+y^2 = 1 \end{cases} \iff \begin{cases} 1 = 2\lambda x \\ 1 = 2\lambda y \\ x^2+y^2=1 \end{cases} \star$$

<u>suppose $\lambda \neq 0$</u>

$\star \Rightarrow x = 1/2\lambda \;, \; y = 1/2\lambda$

putting into constraint:

$$(1/2\lambda)^2 + (1/2\lambda)^2 = 1$$

$$(2\lambda)^2 = 2$$

$$4\lambda^2 = 2$$

$$\lambda^2 = \frac{1}{2}$$

$$\lambda = \pm\sqrt{1/2}$$

$$\lambda = \pm 1/\sqrt{2} \longrightarrow \lambda = \frac{+1}{\sqrt{2}} \text{ OR } \lambda = \frac{-1}{\sqrt{2}}$$

$\lambda = \frac{+1}{\sqrt{2}}$: $x = \frac{1}{2\lambda} = \frac{1}{2(1/\sqrt{2})} = 1/\sqrt{2}$

similarily $y = 1/\sqrt{2}$

so we have a candidate point $(1/\sqrt{2}, 1/\sqrt{2})$

$\lambda = \frac{-1}{\sqrt{2}}$: $x = \frac{1}{2\lambda} = \frac{1}{2(-1/\sqrt{2})} = -1/\sqrt{2}$

similarily $y = -1/\sqrt{2}$

So we have a candidate point $(-1/\sqrt{2}, -1/\sqrt{2})$

<u>If $\lambda = 0$</u>

$1 = 2\lambda x \Rightarrow 1 = 0$

This is nonsense, so $\lambda \neq 0$

Check our values:

$f(1/\sqrt{2}, 1/\sqrt{2}) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \boxed{\sqrt{2}}$ ← max value

$f(-1/\sqrt{2}, -1/\sqrt{2}) = -1/\sqrt{2} + -1/\sqrt{2} = -2/\sqrt{2} = \boxed{-\sqrt{2}}$ ← min value

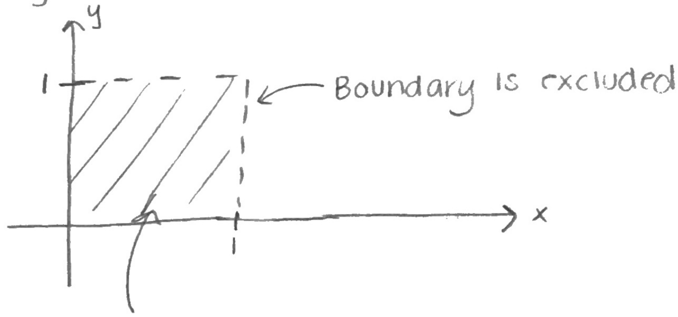
EXTREME VALUE THEOREM

Theorem: Let $R \subseteq \mathbb{R}^2$ which is closed and bounded
Then any continuous function achieves a max and min on R.

We need to explain what these things mean.

example:

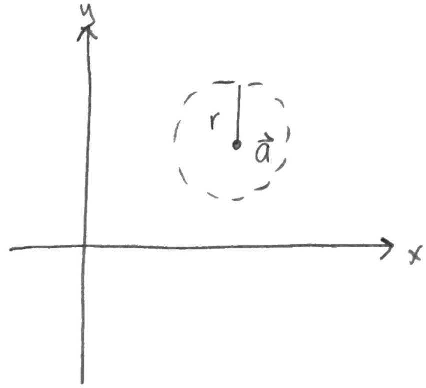
$\{x,y : 0 < x < 1$ and $0 < y < 1\}$



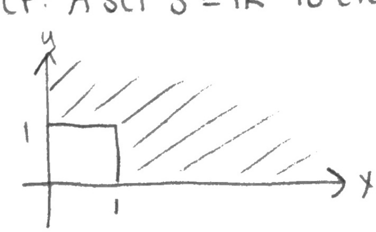
This is an open set

Def: A set $S \subseteq \mathbb{R}^2$ is open if for every $\vec{a} \in S$, there exists an open disc centered at $\vec{a}$ completely confined in S.

$\left(\begin{array}{c}\text{Open disc centered at } \vec{a} \\ \text{of radius } r\end{array}\right) = B_r(\vec{a})$
$= \{\vec{x} \in \mathbb{R}^2 : |\vec{x} - \vec{a}| < r\}$



Def: A set $S \subseteq \mathbb{R}^2$ is closed if it is the complement of an open set.



$\mathbb{R}^2 \setminus$ previous example
complement

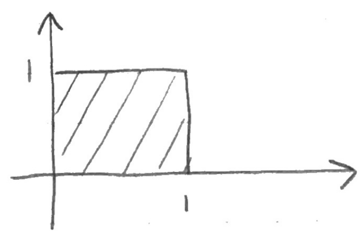
$\{(x,y) : \text{not } (0 < x < 1 \text{ and } 0 < y < 1)\} = \{(x,y) : x \leq 0 \text{ or } x \geq 1 \text{ or } y \leq 0 \text{ or } y \geq 1\}$

This set is closed but not bounded.

Does have parts going off to ∞

Example: A closed & bounded set ("compact" set)

The set $\{(x,y): 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$ is closed and bounded



example problem: Find the extreme values of

$$f(x,y) = x^2 + y^2$$

on the square

$$\{(x,y): 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$$

How to solve:

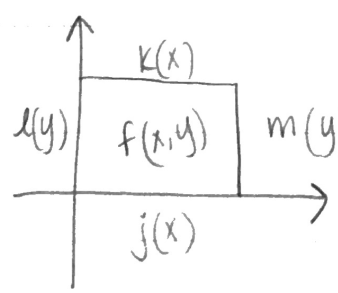
Find local extrema on the interior

$\nabla f = 0$ (throw away values outside the domain)

Find extreme values on the boundary

$j(x) = f(x,0)$ $\quad$ $l(y) = f(0,y)$

$k(x) = f(x,1)$ $\quad$ $m(y) = f(1,y)$



Taking the max (resp. min) of all the values from the first two parts gives your extrema.