$\boxed{\text{LAST TIME:}}$

Wednesday 10/5/16

↳ Extreme value Theorem

↳ Lagrange multipliers

$\boxed{\text{TODAY:}}$ more Lagrange multipliers

example problem:

optimize
$$f(x,y,z) = xyz$$
subject to
$$x+y+z=1$$

3 vars
1 constraint

soln: The solution of Lagrange multiplier problems in more variables is similar to the two variable version.

Step 1 →

solve, $\begin{cases} \nabla f = \lambda \nabla g \\ g = k \end{cases}$ λ is the lagrange multiplier

Step 2 ⟶

Plug the solutions into f and take the max/min of that set

In this particular problem:

$$\nabla f = (yz, xz, xy)$$
$$\nabla g = (1,1,1)$$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = k \end{cases} \iff \begin{cases} (yz, xz, xy) = (1,1,1) \\ x+y+z=1 \end{cases} \iff \begin{cases} yz = \lambda \\ xz = \lambda \\ xy = \lambda \\ x+y+z=1 \end{cases}$$

we solve this system of eqns

Two ways to solve: Trick way & Direct way

TRICK: $\begin{cases} yzx = \lambda x \\ xyz = \lambda y \\ xyz = \lambda z \\ x+y+z=1 \end{cases} \Rightarrow \lambda x = \lambda y = \lambda z$

If $\lambda \neq 0$: $\lambda x = \lambda y \Rightarrow \lambda x - \lambda y = 0$

$\lambda(x-y) = 0$

$\lambda \neq 0 \quad y = x$

similarily, $x = z$

using $x+y+z=1 \Rightarrow 3x = 1 \Rightarrow x = 1/3$

$\Rightarrow (x,y,z) = (1/3, 1/3, 1/3)$

If $\lambda = 0$: $xyz = 0$ (from any one of our eqns)

If $x = 0$: we need to look at what happens to y & z

$y+z=1$

$y = 1-z$

putting this back into $yzx = \lambda x$

$(1-z)z = 0$

$z=0$ OR $z=1$

If $z=0$: and $x=0$ $x+y+z=1 \Rightarrow 0+y+0=1$

$\Rightarrow y=1$

$\Rightarrow (0,1,0)$

If $z=1$: and $x=0$ $x+y+z=1 \Leftrightarrow 0+y+1=1$

$\Leftrightarrow y=0$

$(0,0,1)$

If $x \neq 0$: since $xyz=0$ this means $y=0$ or $z=0$

If $y=0$: and $x \neq 0$ $xz = x = 0$

[we are still in case $\lambda=0$] $\Rightarrow z=0$

$x+y+z=1 \Leftrightarrow x+0+0=1 \Rightarrow x=1$

we get the point $(1,0,0)$

If $z=0$: and $x \neq 0$ we get a similar thing, $(1,0,0)$

Recap:

general

$\lambda = 0$ | $x = y$

$(1/3, 1/3, 1/3)$

$x \neq 0$ | $x = 0$

$(1,0,0)$ | $(0,1,0)$, $(0,0,1)$

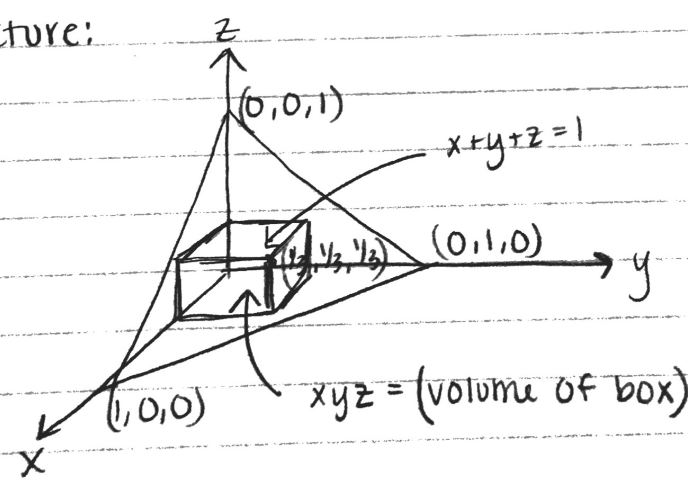
we could have used any variable

Step 2 $\longrightarrow$ plug in values

$f(1/3, 1/3, 1/3) = (1/3)^3 = 1/27$ maximum

$$\left.\begin{aligned} f(1,0,0) &= 0 \\ f(0,1,0) &= 0 \\ f(0,0,1) &= 0 \end{aligned}\right\} \text{minimum}$$

Picture:



example problem:

optimize

$$T(x,y,z) = 20 + 2x + 2y + z^2$$

subject to

$$x^2+y^2+z^2=11$$
$$x+y+z=3$$

This is a lagrange multiplier problem with two constraints

Here we have a function and

$$g(x,y,z) = k_1$$
$$h(x,y,z) = k_2$$

$$g(x,y,z) = x^2+y^2+z^2, \quad k_1 = 11$$
$$h(x,y,z) = x+y+z, \quad k_2 = 3$$

To solve a lagrange multiplier problem with two constraints, we solve

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g = k \\ h = k_2 \end{cases}$$

then we plug solutions back into f and take max/min of those values

Solution

$T = 20 + 2x + 2y + z^2$
$x^2 + y^2 + z^2 = 11$
$x + y + z = 3$

Solution:

$\nabla T = (2, 2, 2z)$
$\nabla g = (2x, 2y, 2z)$
$\nabla h = (1, 1, 1)$

our system of equations is

$$\begin{cases} \nabla T = \lambda \nabla g + \mu \nabla h \\ x^2 + y^2 + z^2 = 11 \\ x + y + z = 3 \end{cases}$$

$$\Leftrightarrow \begin{cases} (2, 2, 2z) = \lambda (2x, 2y, 2z) + \mu (1, 1, 1) \\ x^2 + y^2 + z^2 = 11 \\ x + y + z = 3 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2 = 2\lambda x + \mu & (1) \\ 2 = 2\lambda y + \mu & (2) \\ 2z = 2\lambda z + \mu & (3) \\ x^2 + y^2 + z^2 = 11 & (4) \\ x + y + z = 3 & (5) \end{cases}$$

This is what we need to solve for x, y, z, λ & μ

Subtract 2nd from 1st

$$\begin{cases} 0 = 2\lambda x - 2\lambda y = 2\lambda (x - y) & (1') \\ 2z = 2\lambda z + \mu & (2') \\ x^2 + y^2 + z^2 = 11 & (3') \\ x + y + z = 3 & (4') \end{cases}$$

$(1') = 2\lambda = 0$ or $x - y = 0$

Suppose $x=y$!

$$\begin{cases} 2x^2+z^2=11 \\ 2x+z=3 \end{cases}$$

We can solve this for x & z

$2x+z=3 \Rightarrow z=3-2x$

$\Rightarrow 2x^2+(3-2x)^2=11$

$\Rightarrow 2x^2+9-2(3)(-2)x+4x^2=11$

$\Rightarrow 6x^2+12x-2=0$

$2\neq 0 \Rightarrow 2(3x^2+6x-1)=0$

$\Rightarrow 3x^2+6x-1=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-6\pm\sqrt{36-4(3)(-1)}}{2(3)}$$

$$=\frac{-6\pm\sqrt{12(4)}}{6}=\frac{6\pm4\sqrt{3}}{6}=1\pm\frac{2}{3}\sqrt{3}$$

$x=1+\frac{2}{3}\sqrt{3}$ or $x=1-\frac{2}{3}\sqrt{3}$

if $x=1+\frac{2}{3}\sqrt{3}$: cont....