Homework03 solutions - Dupuy 12/1 2016

Problem 1:

$$\vec{r}(t) = (t^2, t^3, t^4).$$

$$\begin{pmatrix}\text{Arc}\\ \text{Length}\end{pmatrix} = \int_0^1 |\vec{r}\,'(t)|\,dt$$

$$= \int_0^1 \sqrt{(2t)^2 + (3t^2)^2 + (4t^3)^2}\,dt$$

$$= \int_0^1 \sqrt{4t^2 + 9t^4 + 16t^6}\,dt$$

(You can actually do this using integration tables & completing the square but it is a pain).

Problem 2:

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,''(t)|^3}$$

We first compute the parts we need

$\vec{r}\,'(t) = (x'(t), y''(t))$.

$\vec{r}\,''(t) = (x''(t), y''(t))$.

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = (x'(t)y''(t) - y'(t)x''(t))\hat{k}.$$

$$|\vec{r}\,''(t)|^3 = (x'(t)^2 + y''(t)^2)^{3/2}.$$

$$\Longrightarrow \kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,''(t)|^3}$$

$$= \frac{|x'(t)y''(t) - y'(t)x''(t)|}{(x''(t)^2 + y''(t)^2)^{3/2}}. \quad //$$

Problem 3:

$$\vec{\gamma}(t) = \left(\frac{2}{t^2+1} - 1, \frac{2t}{t^2+1} \right)$$

$$\vec{\gamma}(0) = (1, 0).$$

$$x'(t) = \frac{d}{dt}\left[\frac{2}{t^2+1} - 1 \right] = -2\,(t^2+1)^{-2} \cdot (2t)$$

$$= \frac{-4t}{(t^2+1)^2}.$$

$$y'(t) = \frac{d}{dt}\left[\frac{2t}{t^2+1} \right] = \frac{2(t^2+1) - 2t(2t)}{(t^2+1)^2}$$

$$= \frac{2t^2+2-4t^2}{(t^2+1)^2}$$

$$= \frac{2-2t^2}{(t^2+1)^2}.$$

$$|\vec{\gamma}'(t)|^2 = x'(t)^2 + y'(t)^2$$

$$= \left(\frac{-4t}{(t^2+1)^2}\right)^2 + \left(\frac{2-2t^2}{(t^2+1)^2}\right)^2$$

$$= \frac{1}{(t^2+1)^4}\left[(-4t)^2 + (2-2t^2)^2\right]$$

$$= \frac{1}{(t^2+1)^4}\left[16t^2 + 4 - 8t^2 + 4t^4\right]$$

$$= \frac{1}{(t^2+1)^4}\left[4t^4 + 8t^2 + 4\right]$$

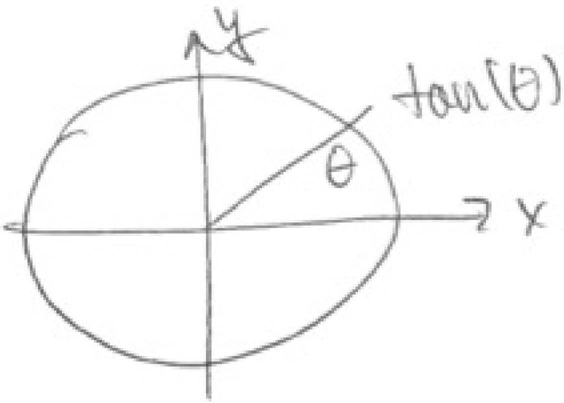
$$= \frac{4}{(t^2+1)^4}\left[t^4 + 2t^2 + 1\right]$$

$$= \frac{4}{(t^2+1)^4}(t^2+1)^2 = \frac{4}{(t^2+1)^2}.$$

$$\Longrightarrow s = \int_0^t |\vec{\gamma}'(\tau)|\,d\tau = \int_0^t \sqrt{\frac{4}{(\tau^2+1)^2}}\,d\tau$$

$$\Rightarrow s = \int_0^t \frac{2}{\tau^2+1}\,d\tau$$

$$= 2\tan^{-1}(\tau)\Big|_{\tau=0}^{\tau=t}$$



$\tan(\theta) = \frac{y}{x} =$ (slope of the line)

$$\Rightarrow s/2 = \tan^{-1}(t) \Rightarrow \tan(s/2) = t.$$

Reparametrizing gives,

$$\vec{p}(s) = \left(\frac{2}{\tan(s/2)^2+1} - 1,\ \frac{2\tan(s/2)}{\tan(s/2)^2+1}\right)$$

We can simplify this,

$$\cancel{\tan(\theta)^2 + 1 = \sec(\theta)^2}$$

$$s\cancel{\ldots}$$

The simplifications require some <u>trig identities</u>.

<u>1st component:</u>

$$\frac{2}{\tan(s/2)^2+1} - 1 = \frac{2}{\sec(s/2)^2} - 1$$

$$= 2\cos(s/2)^2 - 1$$

$$= 2\left(\frac{\cos(s)+1}{2}\right) - 1$$

$$= \cos(s).$$

<u>2nd component:</u>

$$\frac{2\tan(s/2)}{\tan(s/2)^2+1} = \frac{2\tan(s/2)}{\sec(s/2)^2}$$

$$= 2\,\frac{\sin(s/2)}{\cos(s/2)}\cdot\cos(s/2)^2$$

$$= 2\sin(s/2)\cos(s/2)$$

$$= \sin(s).$$

$\Longrightarrow \vec{p}(s) = (\cos(s), \sin(s))$ and the mystery figure is just the unit circle.

Problem 4:

$$\frac{d}{dt}\left[|\vec{r}(t)|^2\right] = \frac{d}{dt}\left[\vec{r}(t)\cdot\vec{r}(t)\right]$$

$$= 2\,\vec{r}(t)\cdot\vec{r}\,'(t)$$

$$= 0$$

where the last line follows from what was given in the problem.

$$\Rightarrow \frac{d}{dt}\left[|\vec{r}(t)|^2\right] = 0$$

$$\Rightarrow \boxed{|\vec{r}(t)|^2 = C}\ .//$$

Writing this out gives,

$$x(t)^2 + y(t)^2 + z(t)^2 = C.$$