TODAY: more on Tangent Planes & the chain rules

9/27/2016

Let's use our formulas for tangent planes in this special case with

$$g(x,y,z) = z - f(x,y)$$
$$(x_0,y_0,z_0) = (x_0, y_0, f(x_0,y_0))$$

Let's compute

$$\nabla g\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right) = \frac{-\partial f}{\partial x}, \frac{-\partial f}{\partial y}, 1$$

Written out, this is

$$\frac{\partial g}{\partial x}(x_0,y_0,z_0)(x-x_0) + \frac{\partial g}{\partial y}(x_0,y_0,z_0)(y-y_0) + \frac{\partial g}{\partial z}(x_0,y_0,z_0)(z-z_0) = 0$$

Btw, there is a short way to write this,

$$\nabla g(\vec{r}_0)\cdot(\vec{r}-\vec{r}_0) = 0$$

Now use the formula

$$\nabla g(\vec{r}_0)\cdot(\vec{r}-\vec{r}_0) = 0$$
$$\left(-\frac{\partial f}{\partial x}(x_0,y_0), -\frac{\partial f}{\partial y}(x_0,y_0), 1\right)\cdot(x-x_0, y-y_0, z-z_0) = 0$$

$$\Longleftrightarrow \boxed{-\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) - \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) + (z - f(x_0,y_0)) = 0}$$

This is the tangent plane to the graph of a function at a point.

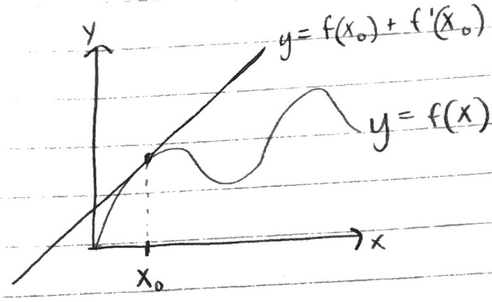
For fun, let's solve for z in this eqn:

$$\boxed{z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}$$

This is the "local linear approximation of the function $f(x,y)$ at the point (x_0,y_0)

In 2D: Recall the formulas for the Taylor Series of a function $f(x)$ at a point x_0:

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{local linear approx}} + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots$$

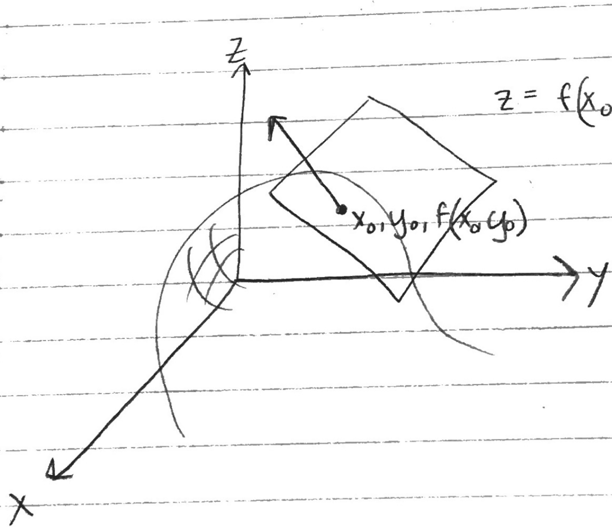


Similarily, in 3D:

$$z = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot (x - x_0, y - y_0)$$

is the equation for the tangent plane

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$



Example: Find the plane tangent to the graph of $f(x,y) = 1 - x^2 - y^2$ at the point $(x_0, y_0, z_0) = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$.

Solutions: 2 ways

general formulas

local linear approximations

$$g(x,y,z) = z - f(x,y)$$

$$\nabla g(\vec{r}_0) \cdot (\vec{r} - \vec{r}_0) = 0$$

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

1. Solution via local linear approx:

$$\frac{\partial f}{\partial x} = -2x \qquad \frac{\partial f}{\partial x}\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = -1 \qquad f\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \tfrac{1}{2}$$

$$\frac{\partial f}{\partial y} = -2y \qquad \frac{\partial f}{\partial x}\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = -1$$

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

$$z = \tfrac{1}{2} + (-1)\left(x - \tfrac{1}{2}\right) + (-1)\left(y - \tfrac{1}{2}\right)$$

$$\Leftrightarrow z \;\; \tfrac{1}{2} - \left(x - \tfrac{1}{2}\right) - \left(y - \tfrac{1}{2}\right) //$$

simplifying...

$$\Leftrightarrow z = -x - y + \tfrac{3}{2} //$$

Both ok

2. Solution via general formula

$$\nabla g(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

$$g(x,y,z) = z - (1 - x^2 - y^2)$$

$$\frac{\partial g}{\partial x} = 2x \qquad \frac{\partial g}{\partial y} = 2y \qquad \frac{\partial g}{\partial z} = 1$$

$(x_0, y_0, z_0) = \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)$

$$\nabla g\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) = (1, 1, 1)$$

$$\nabla g\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) \cdot \left(x - \tfrac{1}{2}, y - \tfrac{1}{2}, z - \tfrac{1}{2}\right) = 0$$

$$\Leftrightarrow (1,1,1) \cdot \left(x - \tfrac{1}{2}, y - \tfrac{1}{2}, z - \tfrac{1}{2}\right) = 0 //$$

$$\Leftrightarrow x - \tfrac{1}{2} + y - \tfrac{1}{2} + z - \tfrac{1}{2} = 0$$

$$\Leftrightarrow x + y + z - \tfrac{3}{2} = 0 //$$

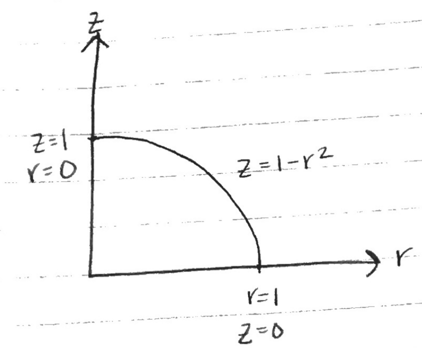
Both ok

Same answers!

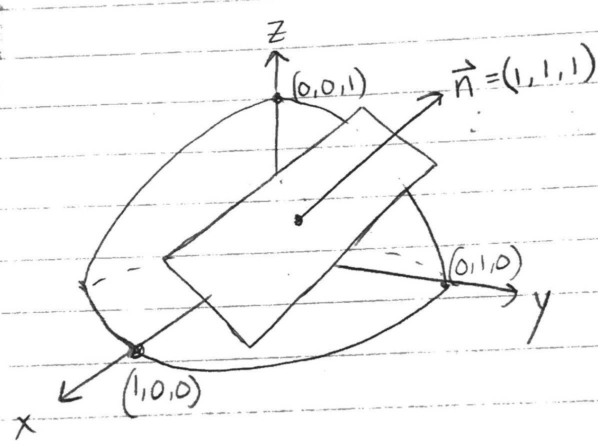
For fun, lets plot the tangent planes!

$x = r\cos\theta$
$y = r\sin\theta$

$$z = f(x,y) = 1-(x^2+y^2) = 1-r^2$$



we take a revolution of this around the z-axis to get our picture.



<u>yz-trace</u>
$z = 1 - y^2$

<u>xy-trace</u>
$1 - x^2 - y^2 = 0$
(a circle)

$\boxed{\text{Summary}}$ 2 Formulas:

setting $g(x,y,z) = z - f(x,y)$

1. $\nabla g(\vec{r}_0)\cdot(\vec{r}-r_0) = 0$ → applies to things like $ze^{z}y + x = 0$ b/c you can't isolate z → This one is more general, USE IT MORE!

2. $z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$..

CHAIN RULE

Recall from calc I:

$$\frac{\partial}{\partial t}\left[f(g(t))\right] = f'(g(t))\cdot g'(t)$$

In calc III, we have similar formulas

$$f = f(x,y)$$

$$\vec{r}(t) = (x(t), y(t))$$

$$\boxed{\frac{\partial}{\partial t}\left[f(\vec{r}(t))\right] = \nabla f(\vec{r}(t))\cdot\vec{r}'(t)}$$ ⇐ chain Rule!

Let's expand this out for fun ☺

$$\frac{\partial}{\partial t}\left[f(x(t), y(t))\right] = \frac{\partial f}{\partial x}(x(t), y(t))\,x'(t) + \frac{\partial f}{\partial y}(x(t), y(t))\,y'(t)$$

From this chain rule (which he hasn't explained yet), we get more rules:

$$x = x(s,t)$$

$$y = y(s,t)$$

$$\frac{\partial}{\partial t}\left[f(x(s,t), y(s,t))\right] = \frac{\partial f}{\partial x}(x(s,t), y(s,t))\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}(x(s,t), y(s,t))\frac{\partial y}{\partial t}$$

Similarly

$$\frac{\partial}{\partial s}\Big[f\big(x(s,t),y(s,t)\big)\Big] = \frac{\partial f}{\partial x}\big(x(s,t),y(s,t)\big)\frac{\partial x}{\partial s} + \frac{\partial t}{\partial y}\big(x(s,t),y(s,t)\big)\frac{\partial y}{\partial s}$$

To make the notation shorter, we omit the stuff we plug in and write

$$\boxed{\begin{aligned} \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \end{aligned}}$$ $\Leftarrow$ Chain Rule

There exists versions of this for many variables

example:

Find $\frac{dz}{dt}$ where, $z = xy^3 - x^2y$

$x = t^2+1$

$y = t^2-1$

Solution:

uses the first version of the chain rule.: $\frac{\partial}{\partial t}\big[f(\vec{r}(t))\big] = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$

$f(x,y) = xy^3 - x^2y$

$$\frac{d}{dt}\big[f(t^2+1, t^2-1)\big] = \frac{\partial f}{\partial x}(t^2+1, t^2-1)[2t] + \frac{\partial f}{\partial y}(t^2+1, t^2-1)[2t]$$

$$\left\{ \frac{\partial f}{\partial x} = y^3 - 2xy \qquad \frac{\partial f}{\partial y} = 3y^2x - x^2 \right\}$$ SIDE WORK

$$\frac{d}{dt}\big[f(t^2+1, t^2-1)\big] = \big[(t^2-1) - 2(t^2+1)(t^2-1)\big]2t + \big[3(t^2-1)(t^2+1) - (t^2+1)^3\big]2t$$

$$= \big[(t^2-1)(t^2-1)^2 - 2(t^4-1)\big]2t$$

will finish next class →