9/28/16 Notes:

- Chain Rules continued:

  - example$_1$: $f = f(x, y)$
    $\vec{r}(t) = (x(t), y(t))$

$$\frac{d}{dt}\left[f(\vec{r}(t))\right] = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = \frac{\partial f}{\partial x}(\vec{r}(t))x'(t) + \frac{\partial f}{\partial y}(\vec{r}(t))y'(t)$$

  - example$_2$: $\vec{r}(s,t) = (x(s,t), y(s,t))$
    $f = f(x,y)$

$$\boxed{\begin{aligned} \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \end{aligned}}$$

- Example problem:
  compute $\frac{d}{dt}\left[F(\vec{r}(t))\right]$ where...

$$f(x,y) = xy \quad \& \quad \vec{r}(t) = (t^2, t^3)$$

Solution:

(old way) $f(\vec{r}(t)) = f(x(t), y(t)) = x(t)y(t)$

(chain rule) $\frac{d}{dt}\left[F(\vec{r}(t))\right] = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$

$= (y(t), x(t)) \cdot (2t, 3t^2)$

$= (t^3, t^2)(2t, 3t^2) = 2t^4 + 3t^4 \boxed{= 5t^4}$ //

- Example problem: $w = 2xy$, $x = s^2 + t^2$, $y = \frac{s}{t}$

  compute: $\frac{\partial w}{\partial s}$ & $\frac{\partial w}{\partial t}$

(without chain rule) $\rightarrow$ $w = 2xy = 2(s^2+t^2)\left(\frac{s}{t}\right) \rightarrow$

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$$2(s^2+t^2)\left(\frac{s}{t}\right) = \frac{2s^3}{t} + 2st$$

$$\frac{\partial w}{\partial s} = \frac{\partial}{\partial s}\left[\frac{2s^3}{t} + 2st\right] = \frac{6s^2}{t} + 2t$$

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left[\frac{2s^3}{t} + 2st\right] = \frac{-2s^3}{t^2} + 2s \quad //$$

(with chain rule) → $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$

$$= (2y)(2s) + (2x)\left(\frac{1}{t}\right)$$

$$= 2\left(\frac{s}{t}\right)(2s) + 2(s^2+t)^2\left(\frac{1}{2}\right)$$

$$= \frac{4s^2}{t} + \frac{2s^2}{t} + 2t$$

$$\boxed{= \frac{6s^2}{t} + 2t}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

$$= (2y)(2t) + (2x)\left(-\frac{s}{t^2}\right)$$

$$= 2\left(\frac{s}{t}\right)(2t) + 2(s^2+t^2)\left(-\frac{s}{t^2}\right) = 4s - \frac{2s^3}{t^2} - 2st$$

$$\boxed{= 2s - \frac{2s^3}{2t}} \quad //$$

- General Chain Rule:

If $f = f(x_1, x_2, x_3, \ldots x_n)$ and $x = x_1(t_1, t_2 \ldots t_m)$

$x_n = X_n(t_1, t_2, t_3 \ldots t_m)$

$$\boxed{\frac{\partial f}{\partial t_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial t_i}}$$

$$\boxed{= \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_i}} \quad //$$