Final Exam — Dupuy — Math 121 — Fall 2016

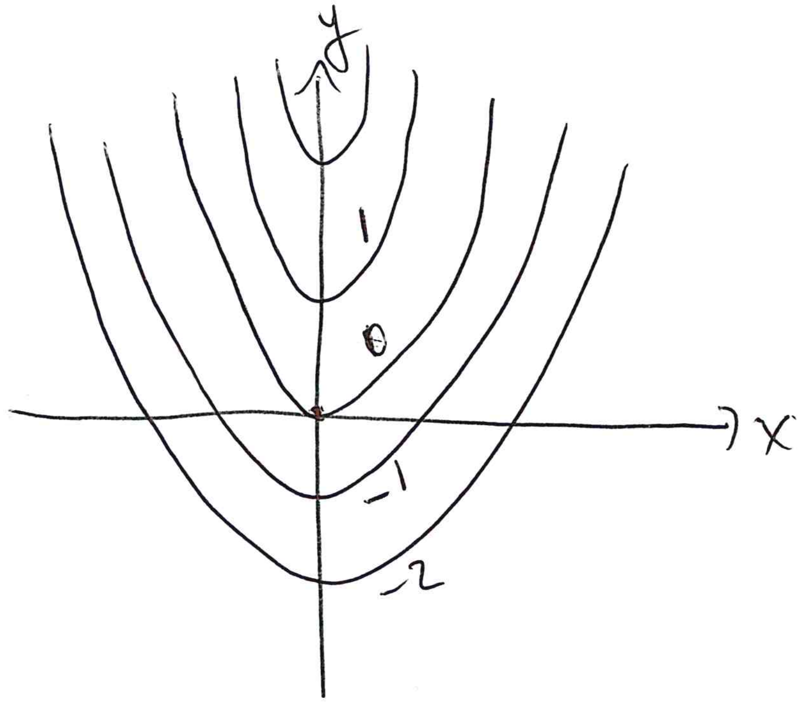
Instructions Remember to show all your work to get full credit. Please leave answers in their exact form. This is a closed book test. You may not use a calculator. If you need extra paper let me know. You will be marked off for floating expressions and equalities not related to the problem. You can potentially be marked off for being vague or imprecise.

Name: Key

Section: 121A

Problem	Possible	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
EC1	10	
EC2	10	
Total	120	
Percentage		

1. Make a level set plot of the function $f(x, y) = y - x^2$.



2. Compute the following directions derivatives. **If the direction is not a unit vector, normalize it.**

   (a) $(D_{\vec{u}}f)(1,1)$ where $f(x,y) = x^2 - y^2$ and $\vec{u} = (0,1)$.

   (b) $(D_{\vec{v}}g)(0,0,0)$ where $g(x,y,z) = xe^{xy}$ and $\vec{v} = (1,1,1)$.

a) $\nabla f = (2x, -2y)$

$\vec{u} = (0,1)$

$(D_{\vec{u}}f)(1,1) = \nabla f(1,1) \cdot \vec{u} = -2$

b) $\nabla g = (e^{xy} + xye^{xy}, x^2e^{xy}, 0)$

$\nabla g(0,0,0) = (1,0,0)$

$(D_{\vec{v}}g)(0,0,0) = \nabla g(0,0,0) \cdot \frac{\vec{v}}{|\vec{v}|}$

$= (1,0,0) \cdot \frac{(1,1,1)}{\sqrt{3}} = \frac{1}{\sqrt{3}}.$

3. Find a parametrization of the line passing through the points $(0, 1, 2)$ and $(1, 1, 0)$.

$$\vec{r}(t) = (0,1,2)t + (1,1,0)(1-t)$$

$$= (0,t,2t) + (1-t,1-t,0)$$

$$= (1-t,\ t+(1-t),\ 2t)$$

$$= (1-t,\ 1,\ 2t). \,//$$

4. Let $\vec{a} = (1, 1, 0)$ and $\vec{b} = (0, 1, 1)$

(a) Compute $\vec{a} \cdot \vec{b}$.

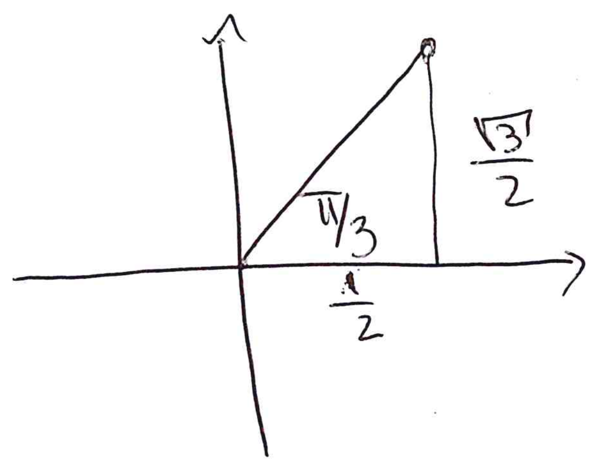
(b) Find the angle between $\vec{a}$ and $\vec{b}$.

(a) $\vec{a} \cdot \vec{b} = (1,1,0) \cdot (0,1,1) = 1.$

(b) $\cos(\theta) = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{1}{\sqrt{2}\,\sqrt{2}} = \dfrac{1}{2}.$

$\theta = \cos^{-1}(1/2)$

$= \dfrac{\pi}{3}$ or 60°



5. Find the line tangent to the curve $\vec{\beta}(t) = (e^{3t}, e^t, e^{2t})$ at the point $(1, 1, 1)$.

$\vec{\beta}(0) = (1,1,1)$

$\vec{\beta}'(t) = (3e^{3t}, e^t, 2e^{2t})$

$\vec{\beta}'(0) = (3,1,2)$.

$\vec{l}(\tau) = (1,1,1) + \tau(3,1,2)$.

6. Find the plane tangent to the surface $x^4 + y^3 + z^2 = 3$ at the point $(1, 1, 1)$.

$$\nabla g = (4x, 3y, 2z)$$

$$\nabla g(1,1,1) = (4,3,2) = \vec{n}$$

<u>Plane Eqn</u>: $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

$$\Longleftrightarrow (4,3,2) \cdot (x-1, y-1, z-1) = 0$$

$$\Longleftrightarrow 4(x-1) + 3(y-1) + 2(z-1) = 0. \quad //$$

7. Find an equation of the plane containing the lines $(t, 0, 3t-1)$, $(2t, 3t, -1)$.

$\vec{v}_1 = (1,0,3)$

$\vec{v}_2 = (2,3,0)$

normal $= \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{vmatrix}$

$$= i\begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} - j\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} + k\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix}$$

$= -9i + 6j + 3k = (-9,6,3) = \vec{n}$.

lines intersect when $t=0$ on both.

$\vec{\alpha}(0) = (0,0,-1)$, $\vec{\beta}(0) = (0,0,-1)$.

Plane: $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

$(-9,6,3) \cdot (x-0, y-0, z+1) = 0$

$\iff -9x + 6y + 3(z+1) = 0 \iff -3x + 2y + z + 1 = 0$

8. Find and classify the critical points of the function $f(x, y) = xy - x - 1$.

$\nabla f = 0$ 2 solve for C.P.s.

$$\left.\begin{array}{l} 0 = \frac{\partial f}{\partial x} = y - 1 \\ 0 = \frac{\partial f}{\partial y} = x \end{array}\right\} \Longrightarrow \quad y = 1, \ x = 0$$

so $(0,1)$ is a C.P.

<u>2nd Der Test:</u>

$$H(0,1) = \det \begin{bmatrix} f_{xx}(0,1) & f_{xy}(0,1) \\ f_{yx}(0,1) & f_{yy}(0,1) \end{bmatrix}$$

$$= \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 < 0$$

saddle

$\therefore$ only critical pt is $(0,1)$ & it is a saddle.

9. Compute the volume of the region

$$T = \{(x, y, z) : x + y + z \leq 1 \;\&\; x \geq 0 \;\&\; y \geq 0 \;\&\; z \geq 0\}$$

$$\iiint_T dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\,dy\,dx$$

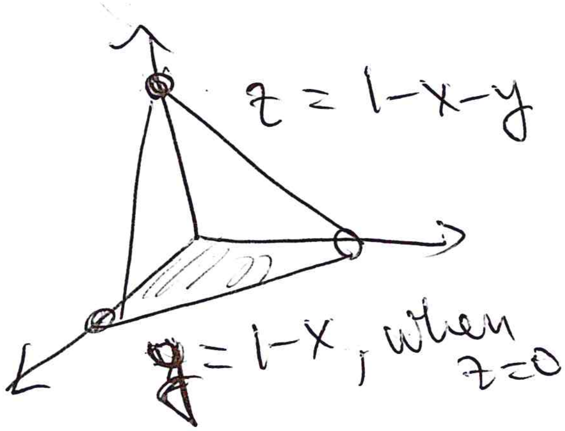
$$= \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$$

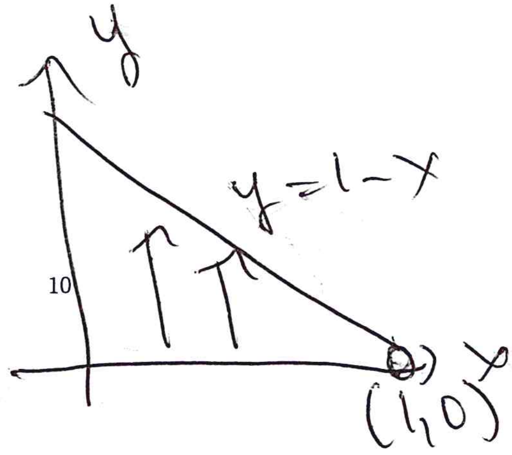
$$= \int_0^1 \cancel{\int_0} \left[(1-x)(1-x) - \frac{(1-x)^2}{2}\right] dx$$

$$= \int_0^1 \frac{(1-x)^2}{2}\,dx = \frac{1}{2}\int_0^1 \frac{d}{dx}\left[-\frac{(1-x)^3}{3}\right] dx$$

$$= -\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)(1-x)^3\Big|_{x=0}^{x=1} = \frac{-1}{6}(0-1) = \frac{1}{6}.$$

Picture:

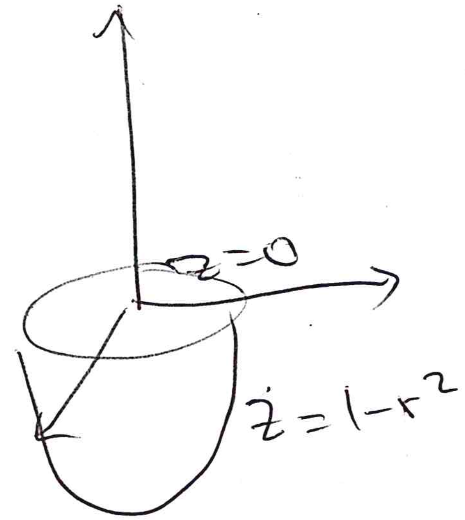




10. Let E be the region below the xy-plane and above the paraboloid $z + 1 = x^2 + y^2$. Compute

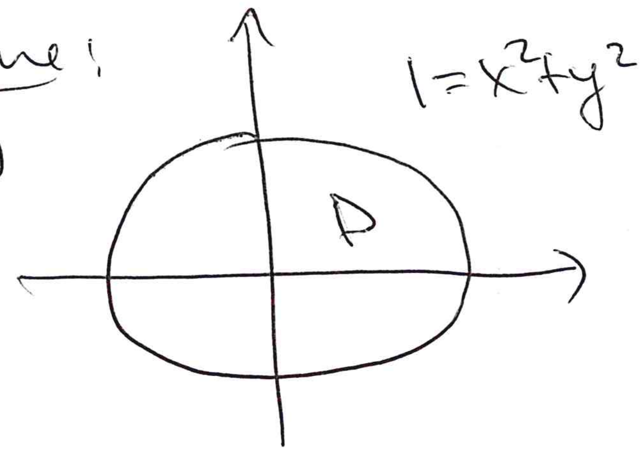
$$\iiint_E z\,dV.$$

picture:



$z \in [1-r^2, 0]$

xy-plane:
$(z=0)$



$r \in [0,1],\ \theta \in [0, 2\pi]$

$$\iiint_E z\,dV = \iint_D \int_{1-r^2}^{0} z\,dz\,dA$$

$$= \iint_D \left.\frac{z^2}{2}\right|_{z=1-r^2}^{z=0} dA$$

$$= \int_0^{2\pi}\int_0^1 -\frac{(1-r^2)^2}{2}\, r\,dr\,d\theta$$

$$= \frac{-1}{2}\left(\int_0^{2\pi} d\theta\right)\underbrace{\left(\int_0^1 (1-r^2)^2 r\,dr\right)}_{\text{computed on back.}} = -\pi\left(\frac{1}{6}\right) = \frac{-\pi}{6}.$$

for problem 10 :

$$\int_0^1 (1-r^2)^2 \, r\,dr$$

$$u = 1-r^2$$
$$du = -2r\,dr$$
$$\Rightarrow \frac{du}{-2} = r\,dr$$

$$\| $$

$$\int_1^0 u^2 \left(\frac{du}{-2}\right)$$

change of bds :

$r=0 \Rightarrow u=1$

$r=1 \Rightarrow u=0.$

$$= \frac{-1}{2}\left(\frac{u^3}{3}\right)\Big|_1^0$$

$$= \frac{-1}{2}\left(\frac{-1}{3}\right) = \frac{1}{6}.$$

11. Let $M = \{(x,y,z) : x^2 + y^2 + z^2 e^z = e^z \ \& \ z \geq 0\}$ with an upward pointing normal. Compute

$$\iint_M \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

Where $\mathbf{F} = \frac{1}{2}(-y, x, z)$. (This surface essentially looks like an upper-half sphere)

Stokes:

$$\iint_M \nabla \times \vec{F} \cdot d\vec{S} = \int_{\partial M} \vec{F} \cdot d\vec{r}$$

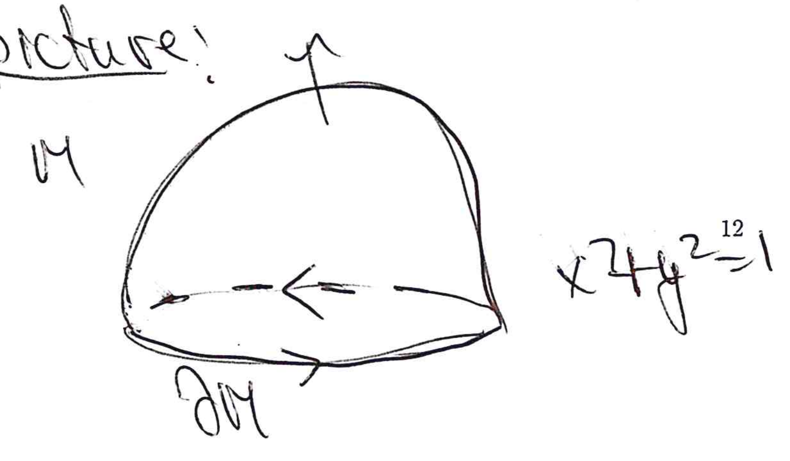
$$= \frac{1}{2} \int_{\partial M} -y\,dx + x\,dy + \cancel{z\,dz}^{\,=0}$$ (in xy-plane $z=0$)

Green's Thm

$$= \frac{1}{2} \iint_D \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\, dA = \iint_D dA$$

$$= \pi(1)^2 = \pi.//$$

picture:



12. Compute the flux of $\mathbf{F} = (x, y^2, z)$ through the *boundary* of the region given by

$$E = \{(x,y,z) : 0 \le z \le 1 - x^2 - y^2\}.$$

$$\iiint_E \nabla \cdot \vec{F}\, dV = \iiint_E (2 + 2y)\, dV$$

$$= \int_0^{2\pi}\int_0^1\int_0^{1-r^2} (2 + 2\sin(\theta) r)\, r\, dz\, dr\, d\theta$$

$$= 2\int_0^{2\pi}\int_0^1 (r + r^2\sin(\theta))(1-r^2)\, dr\, d\theta$$

$$= 2\int_0^{2\pi}\int_0^1 (r + r^2\sin(\theta) - r^3 - r^5\sin(\theta))\, dr\, d\theta$$

$$= 2\int_0^{2\pi} \left(\frac{r^2}{2} + \frac{r^3}{3}\sin(\theta) - \frac{r^4}{4} - \frac{r^6}{6}\sin(\theta)\right)\Big|_{r=0}^{r=1} d\theta$$

$$= 2\int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{3}\sin(\theta) - \frac{1}{4} - \frac{1}{6}\sin(\theta)\right) d\theta$$

$$= 2\left(\frac{1}{2} - \frac{1}{4}\right)(2\pi) = (2-1)\pi = \pi. //$$

EC1 Let M be a simple closed oriented surface in $\mathbf{R}^3$. Prove Stoke's Theorem:

$$\iint_M \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial M} \mathbf{F} \cdot d\mathbf{r}.$$

EC2 Let $C \subset \mathbb{R}^3$ be an oriented curve starting at $\mathbf{a} \in \mathbb{R}^3$ and ending at $\mathbf{b} \in \mathbb{R}^3$. Let $f = f(x, y, z)$ be a scalar function of three variables. Prove the Fundamental Theorem of line integrals:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a})$$