Problem 01:

(a) Show $w = f(x,y)$ satisfies

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cos(\theta) - \frac{\partial w}{\partial \theta}\frac{\sin(\theta)}{r}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\sin(\theta) + \frac{\partial w}{\partial \theta}\frac{\cos(\theta)}{r}$$

where $x = r\cos(\theta)$, $y = r\sin(\theta)$.

Solution

$$\begin{cases} \theta = \tan^{-1}(y/x) \\ r = \sqrt{x^2+y^2} \end{cases}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}\frac{\partial}{\partial x}\left(y/x\right) = \frac{-y}{x^2+y^2}.$$

$$\frac{\partial r}{\partial x} = \frac{1}{2}\left(x^2+y^2\right)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+y^2}}.$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+(y/x)^2}\frac{\partial}{\partial y}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}.$$

$$\frac{\partial r}{\partial y} = \frac{1}{2}\left(x^2+y^2\right)^{-1/2}(2y) = \frac{y}{\sqrt{x^2+y^2}}.$$

Using the chain rule:

$$\begin{aligned}
\frac{\partial w}{\partial x} &= \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} \\
&= \frac{\partial w}{\partial \theta}\left(\frac{-y}{x^2+y^2}\right) + \frac{\partial w}{\partial r}\left(\frac{x}{\sqrt{x^2+y^2}}\right) \\
&= \frac{\partial w}{\partial \theta}\left(-\frac{r\sin(\theta)}{r^2}\right) + \frac{\partial w}{\partial r}\left(\frac{r\cos(\theta)}{r}\right) \\
&= \frac{\partial w}{\partial r}\cos(\theta) - \frac{\partial w}{\partial \theta}\frac{\sin(\theta)}{r}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial w}{\partial y} &= \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial y} \\
&= \frac{\partial w}{\partial r}\left(\frac{y}{\sqrt{x^2+y^2}}\right) + \frac{\partial w}{\partial \theta}\left(\frac{x}{x^2+y^2}\right) \\
&= \frac{\partial w}{\partial r}\left(\frac{r\sin(\theta)}{r}\right) + \frac{\partial w}{\partial \theta}\left(\frac{r\cos(\theta)}{r^2}\right) \\
&= \frac{\partial w}{\partial r}\sin(\theta) + \frac{\partial w}{\partial \theta}\frac{\cos(\theta)}{r}. \quad //
\end{aligned}$$

(There is a 2nd way of doing this which is described on Piazza)

(b) Show

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2.$$

Solution

$$\begin{aligned}
\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 &= \left(\frac{\partial w}{\partial r}\cos(\theta) - \frac{\partial w}{\partial \theta}\frac{\sin(\theta)}{r}\right)^2 + \left(\frac{\partial w}{\partial r}\sin(\theta) + \frac{\partial w}{\partial \theta}\frac{\cos(\theta)}{r}\right)^2 \\
&= \left(\frac{\partial w}{\partial r}\right)^2\cos(\theta)^2 \underline{- 2\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}\cos(\theta)\frac{\sin(\theta)}{r}} + \left(\frac{\partial w}{\partial \theta}\right)^2\frac{\sin(\theta)^2}{r^2} \\
&\quad + \left(\frac{\partial w}{\partial r}\right)^2\sin(\theta)^2 \underline{+ 2\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}\sin(\theta)\frac{\cos(\theta)}{r}} + \left(\frac{\partial w}{\partial \theta}\right)^2\frac{\cos(\theta)^2}{r^2} \\
&= \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2. \quad //
\end{aligned}$$

Problem 02:

(a) $$f(tx,ty) = (tx)^3 - 3(tx)(ty)^2 + (ty)^3$$

$$= t^3x^3 - 3t^3xy^2 + t^3y^3$$

$$= t^3(x^3 - 3x^2y^2 + y^3)$$

$$= t^3 f(x,y).$$

$d = 3.$

(b)

$$g(tx,ty) = t^d g(x,y)$$

$$\Rightarrow \frac{\partial}{\partial t}\left[g(tx,ty)\right] = \frac{\partial}{\partial t}\left[t^d g(x,y)\right]$$

$$\Rightarrow \frac{\partial g}{\partial x}(tx,ty)\frac{\partial}{\partial t}[tx] + \frac{\partial g}{\partial y}(tx,ty)\frac{\partial}{\partial t}[ty] = d\,t^{d-1}g(x,y)$$

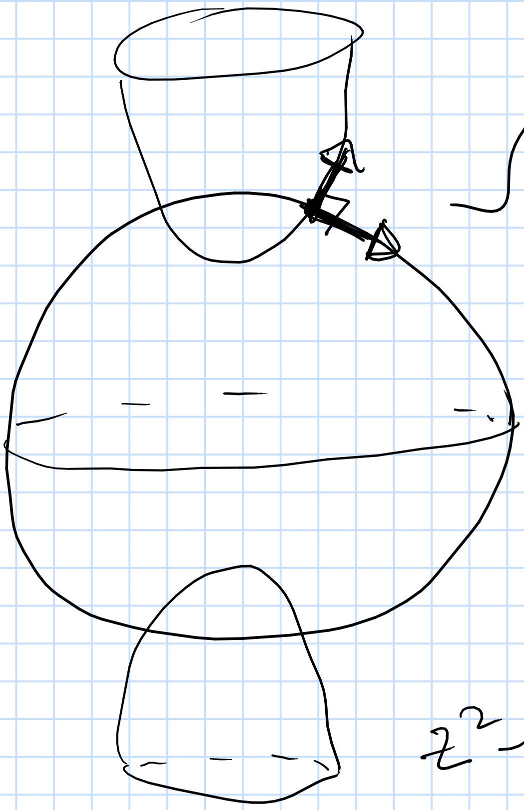
$$\Rightarrow \frac{\partial g}{\partial x}(tx,ty)x + \frac{\partial g}{\partial y}(tx,ty)y = d\,t^{d-1}g(x,y)$$

plug in $t=1$,

$$\frac{\partial g}{\partial x}(x,y)x + \frac{\partial g}{\partial y}(x,y)y = d\,g(x,y). //$$

Problem 03:

You need to show the normal vectors are perpendicular



at the points of intersection of the normal vectors are orthogonal.

$$x^2+y^2+z^2-R^2=0$$
$$\underbrace{x^2+y^2+z^2}_{g(x,y,z)}$$

$$z^2-(x^2+y^2)=0$$
$$\underbrace{z^2-(x^2+y^2)}_{h(x,y,z)}$$

$$\nabla g(x,y,z) = (2x,2y,2z)$$
$$\nabla h(x,y,z) = (-2x,-2y,2z).$$

One needs to evaluate

$$\nabla g \cdot \nabla h$$

at points of intersection.

$$\nabla g(x,y,z)\cdot \nabla h(x,y,z)$$
$$= (2x,2y,2z)\cdot(-2x,-2y,2z)$$
$$= -4x^2-4y^2+4z^2$$
$$= 4(z^2-x^2-y^2)$$
$$=0$$

since (x,y,z) satisfy the equation of the two sheeted hyperboloid.