Problem 01 Dupuy-HW07

A)

$$\int_0^\infty \int_0^\infty e^{-xy} \sin(x)\,dy\,dx$$

$$= \int_0^\infty \sin(x) \int_0^\infty e^{-xy}\,dy\,dx$$

$$= \int_0^\infty \sin(x) \int_0^\infty \frac{d}{dy}\left[-\frac{e^{-xy}}{x}\right] dy\,dx$$

$$= \int_0^\infty \sin(x) \left(-\frac{e^{-xy}}{x} \Big|_{y=0}^{y=\infty} \right) dx$$

$$= \int_0^\infty \sin(x) \left[-0 + \frac{1}{x} \right] dx$$

$$= \int_0^\infty \frac{\sin(x)}{x}\,dx.$$

B)

$$\int_0^\infty \int_0^\infty e^{-xy} \sin(x)\, dy\, dx$$

$$= \int_0^\infty \int_0^\infty e^{-xy} \sin(x)\, dx\, dy$$

Work on inner part,

$u = e^{-xy}$, $dv = \sin(x)dx$

$$\int_0^\infty e^{-xy} \sin(x)\, dx$$

$du = -ye^{-xy}dx$ $\quad v = -\cos(x)$

$$= e^{-xy}(-\cos(x))\Big|_{x=0}^{x=\infty} - \int_0^\infty -ye^{-xy}(-\cos(x))\, dx$$

$$= 1 - y\int_0^\infty e^{-xy} \cos(x)\, dx$$

$$\int_0^\infty e^{-xy} \cos(x)\, dx$$

$$= e^{-xy}(+\sin(x))\Big|_{x=0}^{x=\infty} - \int_0^\infty -ye^{-xy} \sin(x)\, dx$$

$$= 0 + y\int_0^\infty e^{-xy} \sin(x)\, dx$$

$$\int_0^\infty e^{-xy}\sin(x)\,dx = 1 - y^2\int_0^\infty e^{-xy}\sin(x)\,dx$$

$$\Rightarrow \quad A = 1 - y^2 A, \quad A = \int_0^\infty e^{-xy}\sin(x)\,dx$$

$$\Rightarrow \quad A = \frac{1}{1+y^2}.$$

$$\therefore \int_0^\infty\int_0^\infty e^{-xy}\sin(x)\,dx\,dy$$

$$= \int_0^\infty \frac{1}{1+y^2}\,dy$$

$$= \tan^{-1}(\infty) - \tan^{-1}(0) = \frac{\pi}{2}.$$

c)

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \int_0^\infty\int_0^\infty e^{-xy}\sin(x)\,dy\,dx$$

$$= \int_0^\infty\int_0^\infty e^{-xy}\sin(x)\,dx\,dy = \frac{\pi}{2}.$$

Problem 02

A)

$$\int_0^\infty\int_0^\infty e^{-x^2-y^2}\,dx\,dy$$

$$= \int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^\infty \frac{d}{dr}\left[-\frac{e^{-r^2}}{2}\right] dr\,d\theta$$

$$= \frac{\pi}{2}\left(-\frac{e^{-r^2}}{2}\Big|_{r=0}^{r=\infty}\right)$$

$$= \frac{\pi}{2}\left(-0-\left(-\frac{1}{2}\right)\right) = \frac{\pi}{4}.$$

B)

$$\int_0^{\infty} e^{-x^2}\,dx = I$$

$$I^2 = \left(\int_0^{\infty} e^{-x^2} dx\right)\left(\int_0^{\infty} e^{-y^2} dy\right)$$

$$= \int_0^{\infty}\int_0^{\infty} e^{-x^2-y^2}\,dx\,dy$$

$$= \frac{\pi}{4}$$

$$\Rightarrow I^2 = \frac{\pi}{4} \rightarrow I = \frac{\sqrt{\pi}}{2}$$