DUPUY - HW08 - FALL 2016

$\boxed{\text{Problem 1a}}$

$$\vec{F}'(r,\theta) = \begin{bmatrix} \cosh(\theta) & r\sinh(\theta) \\ \sinh(\theta) & r\cosh(\theta) \end{bmatrix}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r\cosh(\theta)^2 - r\sinh(\theta)^2$$

$$= r.$$

$\boxed{\text{Problem 1b}}$

$$\vec{F}'(\rho,\theta,\phi) = \underbrace{\begin{bmatrix} a\sin(\phi)\cos(\theta) & -a\rho\sin(\phi)\sin(\theta) & a\rho\cos(\phi)\cos(\theta) \\ b\sin(\phi)\sin(\theta) & b\rho\sin(\phi)\cos(\theta) & b\rho\cos(\phi)\sin(\theta) \\ c\cos(\phi) & 0 & -c\sin(\phi) \end{bmatrix}}_{\text{Jacobian Matrix}}$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \det\left(\begin{array}{c}\text{Jacobian}\\ \text{Matrix}\end{array}\right)$$

$$= a\sin(\phi)\cos(\theta)\Big(b\rho\sin(\phi)\cos(\theta)(-c\sin(\phi))\Big)$$

$$+ a\rho\sin(\phi)\sin(\theta)\Big(b\sin(\phi)\sin(\theta)(-c\sin(\phi)) - b\rho\cos(\phi)\sin(\theta)(c\cos(\phi))\Big)$$

$$+ a\rho\cos(\phi)\cos(\theta)\Big(0 - b\rho\sin(\phi)\cos(\theta)(c\cos(\phi))\Big)$$

$$= abc\rho^2\sin(\phi)\Big[-\sin(\phi)^2\cos(\theta)^2 - \sin(\phi)^2\sin(\theta)^2 - \cos(\phi)^2\sin(\theta)^2 - \cos(\phi)^2\cos(\theta)^2\Big]$$

$$= -abc\rho^2\sin(\phi)\Big[\sin(\phi)^2(\cos(\theta)^2+\sin(\theta)^2) + \cos(\phi)^2(\sin(\theta)^2+\cos(\theta)^2)\Big]$$

$$= -abc\rho^2 \sin(\phi)\left[\sin(\phi)^2 + \cos(\phi)^2\right]$$

$$= -abc\,\rho^2 \sin(\phi) \,.//$$

Problem 2

$$\iiint_E dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right| d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} abc\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= abc \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= abc\left(\text{vol}(\text{Unit Ball})\right)$$

$$= abc\left(\frac{4}{3}\pi(1)^3\right) = \frac{4}{3}\pi abc. //$$

$\boxed{\text{Problem 3}}$

$$\left.\begin{aligned}
A &= \iiint_E x^2 \rho(x,y,z)\,dV \\
B &= \iiint_E y^2 \rho(x,y,z)\,dV \\
C &= \iiint_E z^2 \rho(x,y,z)\,dV
\end{aligned}\right\} \text{compute these separately}$$

Note that

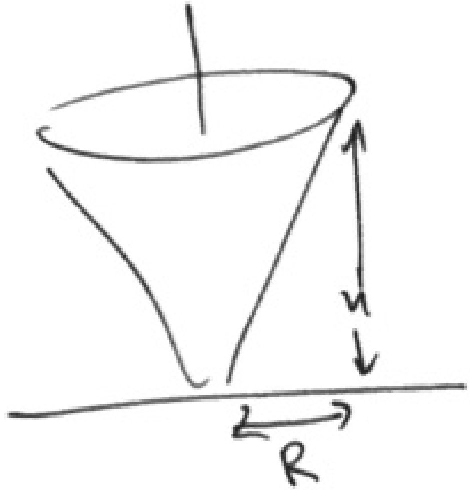
$$I_x = B + C, \quad I_y = A + C, \quad I_z = A + B.$$

we will show

$$\left\{\begin{aligned}
A &= \frac{\pi R^4 h k}{20} \\
B &= \frac{\pi R^4 h k}{20} \\
C &= \frac{\pi R^2 h^2 k}{5}
\end{aligned}\right.$$

Computation for A:

$$\iiint_E x^2 \rho(x,y,z)\, dV$$



$$= k \iiint_E x^2 \, dV$$

$$= k \int_0^{2\pi}\int_0^{R}\int_{\frac{h}{R}r}^{h} r^2 \cos(\theta)^2 r \, dz\, dr\, d\theta$$

$$= k \int_0^{2\pi}\int_0^{R} \left(h - \frac{h}{R} r\right) r^3 \cos(\theta)^2 \, dr\, d\theta$$

$$= kh\left[\left(\int_0^{2\pi}\int_0^{R} r^3 \cos(\theta)^2 \, dr\, d\theta\right) + -\frac{1}{R}\left(\int_0^{2\pi}\int_0^{R} r^4 \cos(\theta)^2 \, dr\, d\theta\right)\right]$$

$$= kh\left[\left(\int_0^{2\pi} \cos(\theta)^2 d\theta\right)\left(\int_0^{R} r^3 dr\right)\right.$$

$$-\frac{1}{R}\left(\int_0^R r^4\,dr\right)\left(\int_0^{2\pi}\cos(\theta)^2\,d\theta\right)\Bigg]$$

$$= kh\left[\left(\frac{\pi}{4}\cdot 4\right)\left(\frac{R^4}{4}\right) - \frac{1}{R}\left(\frac{\pi}{4}\cdot 4\right)\left(\frac{R^5}{5}\right)\right]$$

$$= kh\pi R^4\left(\frac{1}{4}-\frac{1}{5}\right) = kh\pi R^4\left(\frac{5}{20}-\frac{1}{20}\right)$$

$$= \frac{kh\pi R^4}{20}.$$

$$\boxed{A = \frac{kh\pi R^4}{20}}$$

Computation for B:

$$\iiint_E y^2 \rho(x,y,z)\,dV = \text{same}$$

$$\boxed{A = B = \frac{kh\pi R^4}{20}}$$

Computation for C:

$$C = \iiint_E z^2 \rho(x,y,z)\, dV$$

$$= k\int_0^{2\pi}\int_0^R\int_{\frac{h}{R}r}^{h} z^2 \, r \, dz\, dr\, d\theta$$

$$= k\int_0^{2\pi}\int_0^R \left(\frac{h^3}{3} - \frac{1}{3}\left(\frac{h}{R}r\right)^3\right) r \, dr\, d\theta$$

$$= k\cdot 2\pi \int_0^R \left(\frac{h^3 r}{3} - \frac{h^3}{3}\frac{r^4}{R^3}\right) dr$$

$$= k\, 2\pi \left(\frac{h^3}{3}\frac{R^2}{2} - \frac{h^3}{3}\frac{1}{R^3}\frac{R^5}{5}\right)$$

$$\frac{\pi R^2 h^2 k}{5}$$

$$= k(2\pi)\left(\frac{h^3}{3}\right)\left(\frac{R^2}{2} - \frac{R^2}{5}\right)$$

$$= \frac{2\pi k h^3}{3} R^2 \left(\frac{5}{10} - \frac{2}{10}\right) = \boxed{\frac{2\pi k h^3 R^2}{10} = C}$$

Final Computations:

$$I_x = B + C$$

$$= \frac{\pi R^4 h k}{20} + \frac{\pi R^2 h^2 k}{5},$$

$$I_y = A + C$$

$$= (\text{Same as above}) = \frac{\pi R^4 h k}{20} + \frac{\pi R^2 h^2 k}{5},$$

$$I_z = A + B$$

$$= 2\left(\frac{\pi R^4 h k}{20}\right) = \frac{\pi R^4 h k}{10}.$$

Wikipedia:

$$\begin{cases} I_x = I_y = \dfrac{3h}{20}\left(R^2+4h^2\right)m \\ I_z = \dfrac{3hR^2}{10}m \end{cases}$$

Our Answer:

$$\begin{cases} I_x = I_y = \dfrac{\pi R^4 h k}{20} + \pi \dfrac{R^2 h^3}{5} k \\ I_z = \dfrac{\pi R^4 h k}{10} \end{cases}$$

Relationship:

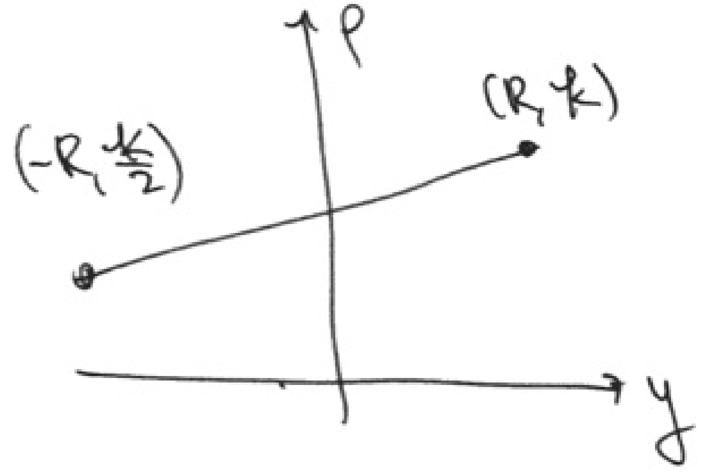
$$\text{Density} = \frac{\text{mass}}{\text{volume}} = \frac{m}{\pi R^2 \frac{h}{3}} = \frac{3m}{\pi R^2 h} = k.$$

$$(\text{our } I_x) = \frac{\pi R^2 h}{20} k \left(R^2 + 4h^2 \right)$$

$$= \frac{\cancel{\pi R^2 h}}{20} \frac{3m}{\cancel{\pi R^2 h}} \left(R^2+4h^2\right)$$

$$= \frac{3m}{20}\left(R^2+4h^2\right) = (\text{wikipedia's } I_x)$$

Problem 4 The density fn is a line so we compute it...



$$\text{slope} = \frac{k - k/2}{R-(-R)} = \frac{k/2}{2R} = \frac{k}{4R}.$$

<u>point-slope:</u>

$$(\rho - k) = \frac{k}{4R}(y-R)$$

$$\Rightarrow \rho = \frac{k}{4R}y - \frac{k}{4} + k = \frac{k}{4R}y + \frac{3k}{4}.$$

$$\rho(x,y,z) = \frac{k}{4R}y + \frac{3k}{4}$$ Density function.

We now compute the mass,

$$m = \iiint_E \rho(x,y,z)\,dV = \star$$

$$\star = \iiint_E \left(\frac{k}{4R}y + \frac{3k}{4}\right) dV$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{R} \left(\frac{k}{4R}\rho\sin(\phi)\sin(\theta) + \frac{3k}{4}\right)\rho^2\sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \frac{k}{4}\int_0^{2\pi}\int_0^{\pi/2}\int_0^{R}\left[\frac{\rho^3\sin(\phi)^2\sin(\theta)}{R} + 3\rho^2\sin(\phi)\right] d\rho\, d\phi\, d\theta$$

$$= \frac{k}{4}\left[A + B\right],$$ where

$$A = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{R}\frac{\rho^3\sin(\phi)^2\sin(\theta)}{R}\, d\rho\, d\phi\, d\theta$$

$$= \frac{1}{R}\left(\int_0^R \rho^3 d\rho\right)\left(\int_0^{\pi/2}\sin(\phi)^2 d\phi\right)\left(\int_0^{2\pi} \cancel{d\theta}\ \sin(\theta) d\theta\right)$$

$$= \frac{1}{R}\left(\frac{R^4}{3}\right)\left(\frac{\pi}{4}\right)\cancel{(2\pi)}(0)$$

$$= 0$$

$$☆ = \iiint_E \left(\frac{k}{4R} y + \frac{3k}{4}\right) dV$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{R} \left(\frac{k}{4R}\, \rho \sin(\phi)\sin(\theta) + \frac{3k}{4}\right) \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \frac{k}{4}\int_0^{2\pi}\int_0^{\pi/2}\int_0^{R} \left[\frac{\rho^3 \sin(\phi)^2 \sin(\theta)}{R} + 3\rho^2 \sin(\phi)\right] d\rho\, d\phi\, d\theta$$

$$= \frac{k}{4}\left[A + B\right],$$ where

$$A = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{R} \frac{\rho^3 \sin(\phi)^2 \sin(\theta)}{R}\, d\rho\, d\phi\, d\theta$$

$$= \frac{1}{R}\left(\int_0^R \rho^3 d\rho\right)\left(\int_0^{\pi/2} \sin(\phi)^2 d\phi\right)\left(\int_0^{2\pi} \cancel{d\theta}\ \sin(\theta) d\theta\right)$$

$$= \frac{1}{R}\left(\frac{R^4}{3}\right)\left(\frac{\pi}{4}\right)\cancel{(2\pi)}\,(0)$$

$$= 0$$

$$B = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{R} 3\rho^2 \sin(\phi)\,d\rho\,d\phi\,d\theta$$

$$= 3\,\frac{4\pi R^3}{6}$$ (because this is just ½ the volume of the sphere)

$$= 2\pi R^3$$

$$\Rightarrow m = \frac{k}{4}(A+B) = \frac{k}{4}(0 + 2\pi R^3)$$

$$= \frac{k\pi R^3}{2}.$$

$$\boxed{m = \frac{k\pi R^3}{2}}$$

<u>Note:</u>

$$\frac{\frac{k}{2}\left(\frac{\pi R^3}{3}\right)}{2} \le m \le \frac{k\left(\frac{\pi R^3}{3}\right)}{2}$$

half density. full density

Moments:

$$m_x = \iiint_E x\, \rho(x,y,z)\, dV$$

$$= \boxed{0} \text{ by symmetry.}$$

$$m_y = \iiint_E y\, \rho(x,y,z)\, dV$$

$$= \iiint_E y\left(\frac{k}{4R}y + \frac{3k}{4}\right) dV$$

$$= \frac{k}{4}\left(\iiint_E \frac{y^2}{R}\, dV + \iiint_E 3y\, dV\right)$$

$$= \frac{k}{4}\left(A + B \right)$$

where

$$A = \iiint_E \frac{y^2}{R}\, dV, \quad B = \iiint_E 3y\, dV.$$

$$A = \iiint \frac{y^2}{R}\, dV$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^{R} \frac{\left(\rho \sin(\phi)\sin(\theta)\right)^2}{R}\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^{R} \frac{\rho^4 \sin(\phi)^3 \sin(\theta)^2}{R}\, d\rho\, d\phi\, d\theta$$

$$= \frac{1}{R}\left[\int_0^{R} \rho^4\, d\rho\right]\left[\int_0^{\pi/2} \sin(\phi)^3\, d\phi \;\; \text{[See sidework]}\right]\left[\int_0^{\pi/2} \sin(\theta)^2\, d\theta\right]$$

$$= \frac{1}{R}\left(\frac{R^5}{5}\right)\left(\frac{2}{3}\right)\left(\frac{\pi}{4}\right)$$

$$= \frac{2\pi R^4}{60} = \frac{\pi R^4}{30}$$

Side work (for m_y)

$$\int_0^{\pi/2} \sin(\phi)^3 d\phi = \int_0^{\pi/2} (1-\cos(\phi)^2)\sin(\phi)d\phi$$

$$= \int_0^{\pi/2} \sin(\phi)\,d\phi - \int_0^{\pi/2} \cos(\phi)^2 \sin(\phi)\,d\phi$$

$$= \int_0^{\pi/2} \frac{d}{d\phi}\left[-\cos(\phi)\right] d\phi$$

$$+ \int_0^{\pi/2} \frac{d}{d\phi}\left[\frac{\cos(\phi)^3}{3}\right] d\phi$$

$$= \cancel{\int_0} -\cos(\phi)\Big|_0^{\pi/2} + \frac{\cos(\phi)^3}{3}\Big|_0^{\pi/2}$$

$$= 1 - 1/3 = \frac{2}{3}.$$

$$B = \iiint_E 3y \, dV$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^R 3\rho \sin(\phi) \sin(\theta) \, \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^R 3\rho^3 \sin(\phi)^2 \sin(\theta) \, d\rho \, d\phi \, d\theta$$

$$= 3 \left(\cancelto{0}{\int_0^{2\pi} \sin(\theta) \, d\phi} \right) \left(\int_0^{\pi/2} \sin(\phi)^2 \, d\phi \right) \left(\int_0^R \rho^3 \, d\rho \right)$$

$$= 0$$

$$\Rightarrow \boxed{\begin{aligned} m_y &= \frac{k}{4} \left(\frac{\pi R^4}{30} + 0 \right) \\ &= \frac{\pi R^4 k}{120} \end{aligned}}$$

$$m_z = \iiint_E z\,\rho(x,y,z)\,dV$$

$$= \iiint_E z\left(\frac{k}{4}\left(\frac{y}{R}+3\right)\right)dV$$

$$= \frac{k}{4}\int_0^{2\pi}\int_0^{\pi/2}\int_0^{R} \rho\cos(\phi)\left(\rho\frac{\sin(\phi)\sin(\theta)}{R}+3\right)\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$

$$= \frac{k}{4}\int_0^{2\pi}\int_0^{\pi/2}\int_0^{R}\left[\rho^4\frac{\cos(\phi)\sin(\phi)^2\sin(\theta)}{R} + 3\rho^3\cos(\phi)\sin(\phi)\right]d\rho\,d\phi\,d\theta$$

$$= \frac{k}{4}(A+B)$$

$$A = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{R} \rho^4 \frac{\cos(\phi)\sin(\phi)^2\sin(\theta)}{R}\, d\rho\, d\phi\, d\theta$$

$$= \frac{1}{R}\left(\int_0^R \rho^4 d\rho\right)\left(\int_0^{\pi/2} \cos(\phi)\sin(\phi)^2 d\phi\right)\cancelto{0}{\left[\int_0^{2\pi} \sin(\theta)\,d\theta\right]}$$

$$= 0.$$

$$B = 3\int_0^{2\pi}\int_0^{\pi/2}\int_0^{R} \rho^3 \cos(\phi)\sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= 3\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/2} \cos(\phi)\sin(\phi)\,d\phi\right)\left(\int_0^R \rho^3 d\rho\right)$$

$$= 3(2\pi)\left(\int_0^{\pi/2} \frac{\sin(2\phi)}{2}\,d\phi\right)\left(\frac{R^4}{4}\right)$$

$$= \frac{3\pi R^4}{2}\left(\int_0^{\pi/2} \frac{d}{d\phi}\left[-\cos(2\phi)\right] d\phi\right)$$

$$= \frac{3\pi}{2}R^4\left(-\cos(\pi)+\cos(0)\right) = \frac{3\pi R^4}{2}.$$

this means

$$m_z = \frac{k}{4}\left(0 + \frac{3\pi R^4}{2}\right)$$

$$= \boxed{\frac{3\pi R^4}{8}k = m_z}$$

We now put everything together to get the center of mass,

$$COM = \frac{1}{m}(m_x, m_y, m_z)$$

$$= \frac{1}{\left(\frac{k\pi R^3}{2}\right)}\left(0, \frac{\pi R^4 k}{120}, \frac{3\pi R^4 k}{8}\right)$$

$$= \frac{2}{k\pi R^3}\left(0, \frac{\pi R^4 k}{120}, \frac{3\pi R^4 k}{8}\right)$$

$$= \boxed{\left(0, \frac{R}{60}, \frac{3R}{4}\right)}$$