Dupuy - Math 121 Fall 2016 - HW09

Problem 1: Plot something in mathematica.

Problem 2:

$$U(\vec{r}) = \frac{Q}{4\pi\varepsilon_0} \frac{-1}{|\vec{r}-\vec{r_0}|}$$

$$\frac{\partial U}{\partial x} = \frac{-Q}{4\pi\varepsilon_0} \frac{\partial}{\partial x}\left[\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} \right]$$

$$= \frac{-Q}{4\pi\varepsilon_0}\left(\frac{-1}{2} \left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2 \right)^{-3/2} \cdot (2(x-x_0)) \right)$$

$$= \frac{+Q}{4\pi\varepsilon_0} \frac{x-x_0}{|\vec{r}-\vec{r_0}|^3}$$

similarly we get,

$$\frac{\partial U}{\partial y} = \frac{Q}{4\pi\varepsilon_0} \frac{y-y_0}{|\vec{r}-\vec{r_0}|^3}, \quad \frac{\partial U}{\partial z} = \frac{Q}{4\pi\varepsilon_0} \frac{z-z_0}{|\vec{r}-\vec{r_0}|^3}$$

So that

$$\nabla U = \frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{k}$$

$$= \frac{Q}{4\pi\varepsilon_0} \frac{1}{|\vec{r}-\vec{r_0}|^3}\left((x-x_0)\hat{\imath} + (y-y_0)\hat{\jmath} + (z-z_0)\hat{k} \right)$$

$$= \underbrace{\frac{Q}{4\pi\varepsilon_0} \frac{1}{|\vec{r}-\vec{r_0}|^3}}_{\text{Coulomb Field.}} (\vec{r}-\vec{r_0})$$

Problem 3

a) $$\nabla \times \nabla f = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix}$$

$$= i\left(\cancel{\partial_y f_z - \partial_z f_y}\right) - j\left(\cancel{\partial_x f_z - \partial_z f_x}\right) + k\left(\cancel{\partial_x f_y - \partial_y f_x}\right)$$

by equality of mixed partials.

$$= 0$$

b) $$\nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) i - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) k$$

$$\nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= \underline{\underline{R_{yx}}} - \underline{\underline{Q_{zx}}} - \underline{\underline{R_{xy}}} + \underline{\underline{P_{zy}}} + \underline{\underline{Q_{xz}}} - \underline{\underline{P_{yz}}}$$

$$= 0.$$

Problem 4 part a

$$\nabla \times (f\vec{F}) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ fP & fQ & fR \end{vmatrix}$$

$$= i\left(\partial_y(fR) - \partial_z(fQ)\right)$$
$$- j\left(\partial_x(fR) - \partial_z(fP)\right)$$
$$+ k\left(\partial_x(fQ) - \partial_y(fP)\right)$$

$$= i\left(\frac{\partial f}{\partial y}R + f\frac{\partial R}{\partial y} - \frac{\partial f}{\partial z}Q - f\frac{\partial Q}{\partial z}\right)$$
$$- j\left(\frac{\partial f}{\partial x}R + f\frac{\partial R}{\partial x} - \frac{\partial f}{\partial z}P - f\frac{\partial P}{\partial z}\right)$$
$$+ k\left(\frac{\partial f}{\partial x}Q + f\frac{\partial Q}{\partial x} - \frac{\partial f}{\partial y}P - f\frac{\partial P}{\partial y}\right)$$

$$= \left[i\left(\frac{\partial f}{\partial y}R - \frac{\partial f}{\partial z}Q\right) - j\left(\frac{\partial f}{\partial x}R - \frac{\partial f}{\partial z}Q\right) + k\left(\frac{\partial f}{\partial x}Q - \frac{\partial f}{\partial y}P\right)\right]$$
$$+ \left[i\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)f - j\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)f + k\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)f\right]$$

$$= \nabla f \times \vec{F} + f(\nabla \times \vec{F}).$$ //

part b

$$\nabla \cdot (f\vec{F})$$
$$= \nabla \cdot (fP\,i + fQ\,j + fR\,k)$$
$$= \frac{\partial}{\partial x}(fP) + \frac{\partial}{\partial y}(fQ) + \frac{\partial}{\partial z}(fR)$$
$$= \frac{\partial f}{\partial x}P + f\frac{\partial P}{\partial x} + \frac{\partial f}{\partial y}Q + f\frac{\partial Q}{\partial y} + \frac{\partial f}{\partial z}R + f\frac{\partial R}{\partial z}$$
$$= \left(\frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + \frac{\partial f}{\partial z}R\right) + f\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)$$
$$= \nabla f \cdot \vec{F} + f(\nabla \cdot \vec{F}).$$