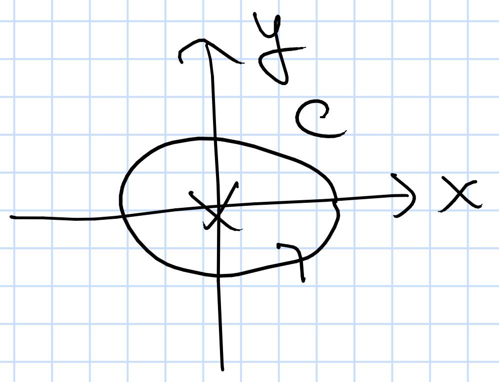
Dupuy - Math 121, Fall 2016 - HW09 - Solutions

Problem 1

(a) $\frac{1}{2\pi}\int_C \frac{x\,dy - y\,dx}{x^2+y^2}$



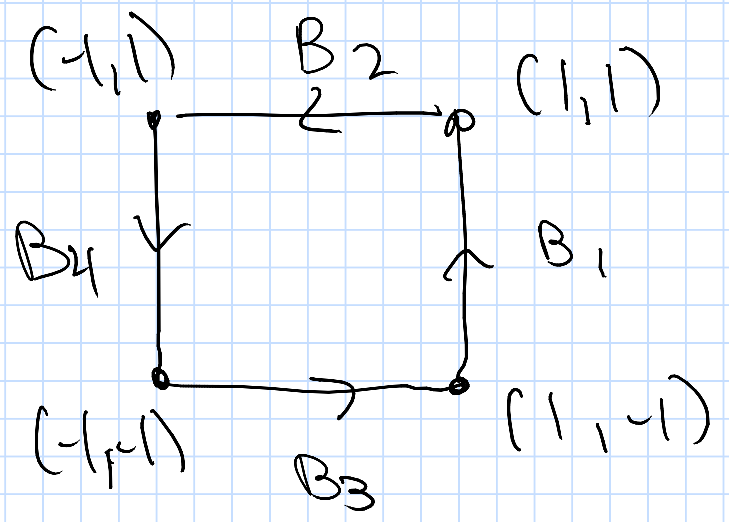
$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos(t)\cos(t)\,dt + \sin(t)\sin(t)\,dt}{1}$

$= \frac{1}{2\pi}\int_0^{2\pi} dt = 1.$

If we go around 3 times:

$$\frac{1}{2\pi}\int_{C_3} \frac{x\,dy - y\,dx}{x^2+y^2} = \frac{3}{2\pi}\int_C \frac{x\,dy - y\,dx}{x^2+y^2} = 3.$$

(b)



$B = B_1 + B_2 + B_3 + B_4$

$$\int_{B_1} \frac{x\,dy - y\,dx}{x^2+y^2} = \int_{-1}^{1} \frac{dy}{1+y^2} = \tan^{-1}(y)\Big|_{y=-1}^{y=1} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.$$

$$\int_{B_2} \frac{x\,dy - y\,dx}{x^2+y^2} = \int_{1}^{-1} \frac{-dx}{x^2+1} = +\int_{-1}^{1} \frac{dx}{x^2+1} = \frac{\pi}{2}.$$

$$\int_{B_3} \frac{x\,dy - y\,dx}{x^2+y^2} = \int_{1}^{-1} \frac{(-1)\,dy}{1+y^2} = +\int_{-1}^{1} \frac{dy}{1+y^2} = \frac{\pi}{2}$$

$$\int_{B_4} \frac{x\,dy - y\,dx}{x^2+y^2} = \int_{-1}^{1} \frac{-(-1)\,dx}{x^2+1} = \int_{-1}^{1} \frac{dx}{x^2+1} = \frac{\pi}{2}.$$

$$\int_B = \int_{B_1} + \int_{B_2} + \int_{B_3} + \int_{B_4} = 2\pi$$

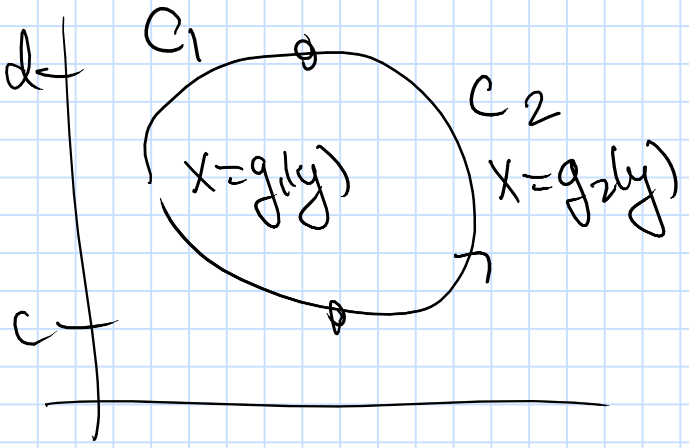
$$\Rightarrow \frac{1}{2\pi}\int_B \frac{x\,dy - y\,dx}{x^2+y^2} = 1.$$

(c) $$\frac{1}{2\pi}\int_D \frac{x\,dy - y\,dx}{x^2+y^2} = \frac{3}{2\pi}\int_B \frac{x\,dy - y\,dx}{x^2+y^2}$$

$$= 3.$$

This example illustrates "winding numbers".

Problem 2



[other people may draw different regions]

$$\int_C Q\,dy = \int_{C_1} Q\,dy + \int_{C_2} Q\,dy$$

$$= \int_d^c Q(g_1(y),y)\,dy + \int_c^d Q(g_2(y),y)\,dy$$

$$= \int_c^d \left[Q(g_2(y),y) - Q(g_1(y),y)\right] dy$$

$$= \int_c^d \int_{g_1(y)}^{g_2(y)} \frac{\partial Q}{\partial x}\,dx\,dy$$

$$= \iint_R \frac{\partial Q}{\partial x}\,dA.$$

(b) We have the equations,

$$\int_C P\,dx = -\iint_R \frac{\partial P}{\partial y}\,dA$$

$$\int_C Q\,dy = \iint_R \frac{\partial Q}{\partial x}\,dA$$

$$\Rightarrow \int_C P\,dx + Q\,dy = -\iint_R \frac{\partial P}{\partial y}\,dA + \iint_R \frac{\partial Q}{\partial x}\,dA$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA,$$

this proves the result. //

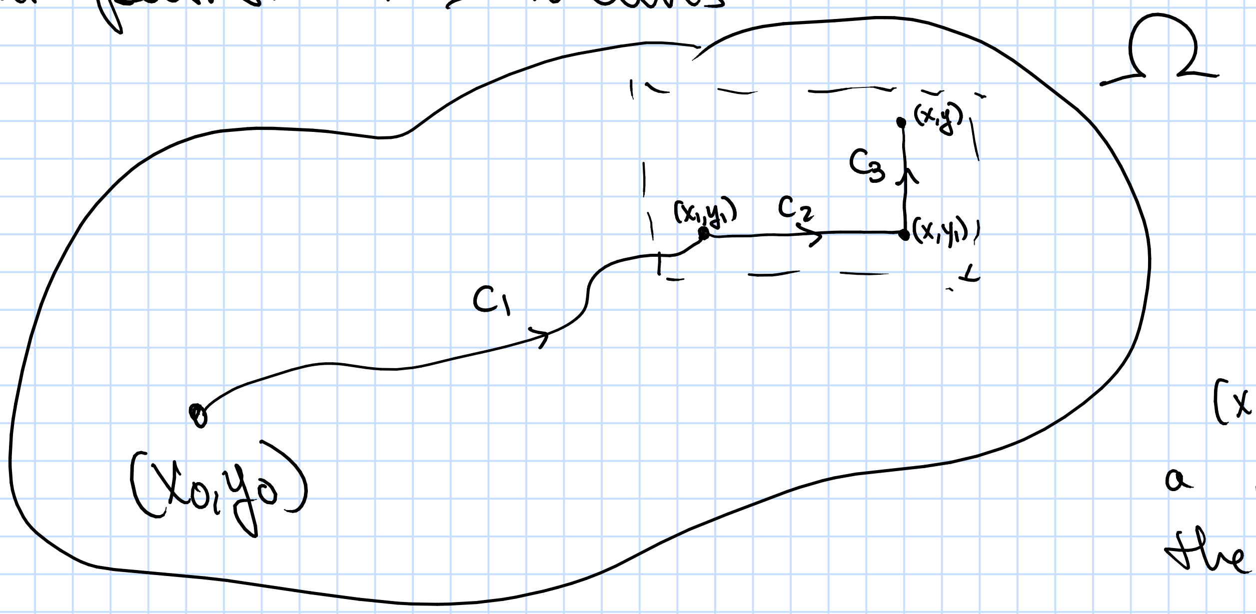
Problem 3

[solutions to this problem will look different for different people]

(a) Let

$$f(x,y) = \int_{(x_0,y_0)}^{(x,y)} \vec{F}\cdot d\vec{r}.$$

By path independence we can choose our paths. This means



Because Ω is open we can find some rectangle containing (x,y) and hence choose a point (x_1,y_1) & form the paths C_1, C_2, C_3.

$$\int_{(x_0,y_0)}^{(x_1,y_1)} P\,dx + Q\,dy = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} (P\,dx + Q\,dy) + \int_{C_3} P\,dx + Q\,dy$$

$$= \underbrace{k}_{\text{some constant}} + \int_{x_1}^{x} P(t,y_1)\,dt + \int_{y_1}^{y} Q(x,t)\,dt$$

$$\Rightarrow \frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}\left[\underbrace{k}_{\text{some constant}} + \int_{x_1}^{x} P(t,y_1)\,dt + \int_{y_1}^{y} Q(x,t)\,dt\right]$$

$$= 0 + \underbrace{0}_{\text{since } \int_{x_1}^{x} P(t,y_1)dt \text{ is just a fn of } x.} + Q(x,y)$$

$\therefore f_y = Q.$ //

(b) From the previous results we had

$$f_x = P$$
$$f_y = Q$$

$$\therefore \nabla f = f_x\hat{i} + f_y\hat{j} = P\hat{i} + Q\hat{j} = \vec{F}.$$ //