Dupuy — Math 121 — Homework 10

Instructions Remember to show all of your work to get credit. Please do this assignment on a separate sheet of paper. Remember to show your work.

1. The following example illustrate the type of phenomena that occur when you don't have a simply connected domain for your vector field but do have a sort-of path independence (up-to winding yourself around the point $(0,0)$). Perform the following computations directly.

   (a) Compute the integral
   $$\frac{1}{2\pi}\oint_C \frac{xdy - ydx}{x^2+y^2},$$
   where C the oriented path starting at $(1,0)$ which goes around the unit circle three times in the counter-clockwise direction.

   (b) Compute the integral
   $$\frac{1}{2\pi}\oint_B \frac{xdy - ydx}{x^2+y^2},$$
   where B is the counter-clockwise oriented closed box with endpoints $(1,-1),(1,1),(-1,1)$ and $(-1,1)$. This is a path made from straight line segments between each of the corners of the box which starts and ends at the point $(-1,1)$.

   (c) Compute the integral
   $$\frac{1}{2\pi}\oint_D \frac{xdy - ydx}{x^2+y^2},$$
   where D is three copies of B attached end-to-end.

2. Let R be the region bounded above by $y = f_2(x)$ and bounded below by $y = f_1(x)$ for $x \in [a,b]$. Let C be the boundary of the region R oriented in the counter-clockwise direction. In class we showed that
   $$\oint_C Pdx = -\iint_R \frac{\partial P}{\partial y} dA. \tag{1}$$

   (a) Doing a similar computation, bounding the region by functions $x \in [g_1(y), g_2(y)]$ and $y \in [c,d]$ show that
   $$\oint_C Qdy = \iint_R \frac{\partial Q}{\partial x} dA. \tag{2}$$
   Make sure to draw the region.

   (b) Conclude Green's Theorem from equations (1) and (2).

3. In class we started the proof that
   $$f(x,y) = \int_{(x_0,y_0)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r}$$
   is a potential for $\mathbf{F}$ provided that line integrals for $\mathbf{F}$ were path independent. We showed that if $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ then $\dfrac{\partial f}{\partial x} = M$.

   (a) Use a similar argument to the one we used in class to show that $\dfrac{\partial f}{\partial y} = N$.

   (b) Conclude that $\nabla f = \mathbf{F}$.