Problem 01

Math 121 - Dupuy - Fall 2016
HW 11 written solutions.

$$\begin{cases} x_\theta = -\frac{v}{2}\sin(\theta/2)\cos(\theta) + (1+v\cos(\theta/2))(-\sin(\theta)) \\ y_\theta = -\frac{v}{2}\sin(\theta/2)\sin(\theta) + (1+v\cos(\theta/2))\cos(\theta) \\ z_\theta = \frac{v}{2}\cos(\theta/2) \end{cases}$$

$$\begin{cases} x_v = \cos(\theta/2)\cos(\theta) \\ y_v = \cos(\theta/2)\sin(\theta) \\ z_v = 0 \end{cases}$$

$\vec{r}_\theta(0,0) = (0,1,0)$, $\vec{r}_v(0,0) = (1,0,0)$

$\vec{r}_\theta(2\pi,0) = (0,1,0)$, $\vec{r}_v(2\pi,0) = (-1,0,0)$

$$\vec{N}(0,0) = (0,1,0)\times(1,0,0)$$
$$= -(0,1,0)\times(-1,0,0)$$
$$= -\vec{N}(2\pi,0)$$

normal vector flips after 2π revolution.

Problem 2

(I told them only to set this up, as answer depends on elliptic integrals)

$x(\theta,\phi) = a\sin(\phi)\cos(\theta)$

$y(\theta,\phi) = b\sin(\phi)\sin(\theta)$

$z(\theta,\phi) = c\cos(\phi)$

$\vec{r}_\theta = (-a\sin(\phi)\sin(\theta), b\sin(\phi)\cos(\theta), 0)$

$\vec{r}_\phi = (a\cos(\phi)\cos(\theta), b\cos(\phi)\sin(\theta), -c\sin(\phi))$

$$\vec{r}_\theta \times \vec{r}_\phi = \begin{vmatrix} i & j & k \\ -a\sin(\phi)\sin(\theta) & b\sin(\phi)\cos(\theta) & 0 \\ a\cos(\phi)\cos(\theta) & b\cos(\phi)\sin(\theta) & -c\sin(\phi) \end{vmatrix}$$

$$= k\left(-ab\sin(\phi)\cos(\phi)\sin(\theta)^2 - ab\sin(\phi)\cos(\phi)\cos(\theta)^2\right) + (-c\sin(\phi))\left[b\sin(\phi)\cos(\theta)i + a\sin(\phi)\sin(\theta)j\right]$$

$$= -c\sin(\phi)^2\left[b\cos(\theta)i + a\sin(\theta)j\right] - ab\sin(\phi)\cos(\phi)k$$

$$|\vec{r}_\theta \times \vec{r}_\phi|^2 = (-c\sin(\phi)^2)^2\left[(b\cos(\theta))^2 + (a\sin(\theta))^2\right] + a^2b^2\sin(\phi)^2\cos(\phi)^2$$

$$= \sin(\phi)^2\left[c^2\sin(\phi)^2\left(b^2\cos(\theta)^2 + a^2\sin(\theta)^2\right) + a^2b^2\cos(\phi)^2\right]$$

$$\Rightarrow \iint_M dS = \underbrace{\int_0^{2\pi}\int_0^{\pi} \sin(\phi)\left[c^2\sin(\phi)\left(b^2\cos(\theta)^2 + a^2\sin(\theta)^2\right) + a^2b^2\cos(\phi)^2\right]^{1/2} d\phi\, d\theta}_{\text{this is some nasty integral you need "elliptic functions" to solve. Sorry for asking this.}}$$

$\shortparallel$

(Surface Area of An Ellipse)

Problem 3

$$\nabla \cdot (f\vec{F}) = \nabla f \cdot \vec{F} + f(\nabla \cdot \vec{F})$$

$\therefore$ $$\iiint_{\Omega} \nabla \cdot (f\vec{F})\, dV = \iiint_{\Omega} \nabla f \cdot \vec{F}\, dV + \iiint_{\Omega} f(\nabla \cdot \vec{F})\, dV$$ (By Product Rule)

$$\iiint_{\Omega} \nabla \cdot (f\vec{F})\, dV = \iint_{\partial\Omega} f\vec{F} \cdot d\vec{S}$$ (By Divergence Thm)

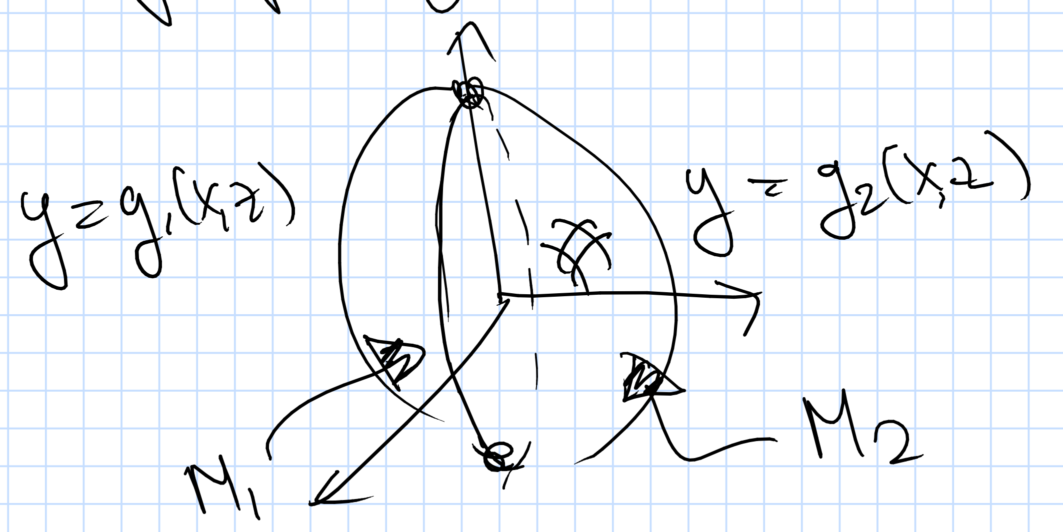
$$\Rightarrow \iint_{\partial\Omega} f\vec{F} \cdot d\vec{S} = \iiint_{\Omega} \nabla f \cdot \vec{F}\, dV + \iiint_{\Omega} f(\nabla \cdot \vec{F})\, dV$$

$$\Rightarrow \boxed{\iiint_{\Omega} f(\nabla \cdot \vec{F})\, dV = \iint_{\partial\Omega} (f\vec{F}) \cdot d\vec{S} - \iiint_{\Omega} \nabla f \cdot \vec{F}\, dV}$$

Problem 4

(a) By a simple region we mean that M is bounded by the graph of two functions:

[I will just do the y-component as the proof for the x-component is similar]



$M = M_2 \cup M_1$

LHS:
$$\iiint_E \frac{\partial Q}{\partial y}\, dV = \iint_D \int_{g_1(x,z)}^{g_2(x,z)} \frac{\partial Q}{\partial y}\, dy\, dA = \iint_D \left[Q(x, g_2(x,z), z) - Q(x, g_1(x,z), z) \right] dA. \quad (*)$$

$dA = dx\,dz$

RHS:
$$\iint_M Q\hat{j}\cdot\hat{N}\,dS$$
$$= \iint_{M_2} Q\hat{j}\cdot\hat{N}\,dS + \iint_{M_1} Q\hat{j}\cdot\hat{N}\,dS$$

Let's work on M_2:
$$\iint_{M_2} Q\hat{j}\cdot\hat{N}\,dS = \iint_D Q(x, g_2(x,z), z)\hat{j}\cdot\left(-\frac{\partial g_2}{\partial x}, 1, -\frac{\partial g_2}{\partial z}\right) dA$$
$$= \iint_D Q(x, g_2(x,z), z)\, dA.$$

for M_1, we get something similar w/ the orientation of the normal vector reversed.
$$\iint_{M_1} Q\hat{j}\cdot\hat{N}\,dS = -\iint_D Q(x, g_1(x,z), z)\, dA$$

$$\Rightarrow \iint_M Q\hat{j}\cdot\hat{N}\,dS = \iint_D \left[Q(x, g_2(x,z), z) - Q(x, g_1(x,z), z) \right] dA. \quad (**)$$

combining $(*)$ & $(**)$ we get
$$\iiint_E \frac{\partial Q}{\partial y}\, dV = \iint_M Q\hat{j}\cdot\hat{N}\,dS.$$

(b) To conclude the proof of divergence we just put them all together:
$$\iiint_E \nabla\cdot\vec{F}\, dV = \iiint_E \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dV$$
$$= \iint_{\partial E} (P\hat{i})\cdot\hat{N}\,dS + \iint_{\partial E} (Q\hat{j})\cdot\hat{N}\,dS + \iint_{\partial E} (R\hat{k})\cdot\hat{N}\,dS$$
$$= \iint_{\partial E} (P\hat{i} + Q\hat{j} + R\hat{k})\cdot\hat{N}\,dS$$
$$= \iint_{\partial E} \vec{F}\cdot\hat{N}\,dS. \;//$$