Dupuy — Math 121 — Homework 11

Instructions Remember to show all of your work to get credit. Please do this assignment on a separate sheet of paper. Remember to show your work.

1. Consider the parametrization of the Möbius strip $\vec{r}(\theta, v) = (x(\theta, v), y(\theta, v), z(\theta, v))$ where

$$\begin{cases} x(\theta, v) = (1 + v\cos(\theta/2))\cos(\theta), \\ y(\theta, v) = (1 + v\cos(\theta/2))\sin(\theta), \\ z(\theta, v) = v\sin(\theta/2), \end{cases}$$

   where $\theta \in [0, 2\pi]$ and $v \in [-1/2, 1/2]$.

   Show that the Möbius strip is not orientable by showing that the normal vector at $\vec{r}(0,0)$ is the negative of the normal vector at $\vec{r}(2\pi, 0)$.

2. Compute the surface area of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

   (Hint: use coordinates from homework 9).

3. Let Ω be a region in $\mathbf{R}^3$ with smooth boundary $\partial\Omega$. Let $\mathbf{F}$ be a vector field and f be a function. Prove the following "integration by parts" formula:

$$\int_\Omega f(\nabla \cdot \mathbf{F})dV = \int_{\partial\Omega} (f\mathbf{F}) \cdot \widehat{\mathbf{N}}dS - \int_\Omega \mathbf{F} \cdot \nabla f dV$$

   (Hint: use the divergence theorem and the product rule for divergence).

4. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. Complete the proof of the divergence theorem for a simple region E with boundary $\partial E = M$. Here are the steps

   (a) Show

$$\iiint_E \frac{\partial P}{\partial x} dV = \iint_M P\mathbf{i} \cdot \widehat{\mathbf{N}}dS,$$
$$\iiint_E \frac{\partial Q}{\partial y} dV = \iint_M Q\mathbf{j} \cdot \widehat{\mathbf{N}}dS.$$

   (b) Conclude the divergence theorem from the equations

$$\begin{cases} \iiint_E \frac{\partial R}{\partial z} dV = \iint_M R\mathbf{k} \cdot \widehat{\mathbf{N}}dS \\ \iiint_E \frac{\partial P}{\partial x} dV = \iint_M P\mathbf{i} \cdot \widehat{\mathbf{N}}dS \\ \iiint_E \frac{\partial Q}{\partial y} dV = \iint_M Q\mathbf{j} \cdot \widehat{\mathbf{N}}dS \end{cases}$$