Midterm 1 — Dupuy — Math 121 — Fall 2016

Instructions Remember to show all your work to get full credit. Please leave answers in their exact form. This is a closed book test. You may not use a calculator. If you need extra paper let me know.

Last Name, First Name:

Section:

Problem	Possible	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	20	
7	20	
8	10	
Total	100	

1. (10 points) Indicate whether the following expressions make sense (Yes or No).

(a) $|\vec{a}| \cdot \vec{b}$

No

(b) $(\vec{a} \cdot \vec{b}) \times \vec{c}$

No

(c) $(\vec{a} \times \vec{b}) \times \vec{c}$

Yes

(d) $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$

No

(e) $\vec{a}/|\vec{a}|$.

Yes

2. (10 points) Consider the vectors

$$\mathbf{a} = \mathbf{i} + \mathbf{j} - 2\mathbf{k},$$

$$\mathbf{b} = \mathbf{i} + \mathbf{j}.$$

Compute the following

(a) $\mathbf{a} \cdot \mathbf{b}$.

$$(i+j-2k)\cdot(i+j) = 1+1 = 2$$

(b) $\mathbf{a} \times \mathbf{b}$.

$$(i+j-2k)\times(i+j) = (i+j)\times(i+j) - 2k\times(i+j)$$
$$= 0 - 2k\times i - 2k\times j$$
$$= -2j + 2i$$

(c) $\text{proj}_{\mathbf{b}}(\mathbf{a})$.

$$\text{proj}_{\vec{b}}(\vec{a}) = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b} = \frac{2}{2}(i+j) = i+j.$$

3. (10 points) Find a parametrization of the line which is the intersection of the two planes:

$$\begin{cases} x - 2y + z = 0, \\ x + y - 2z = 0 \end{cases}$$

$x - 2y + z = 0$

$2x + 2y - 4z = 0$

$\overset{\text{add}}{\Rightarrow} 3x - 3z = 0$

$\Rightarrow x = z.$

back into 2nd: $z + y - 2z = 0$

$\Leftrightarrow y - z = 0 \Rightarrow y = z.$

The line is then

$\ell(t) = (t, t, t).$ //

4. (10 points) Find the line tangent to the curve parametrized by

$$\mathbf{h}(t) = (e^t + 1)\mathbf{i} + (e^{2t} + 2)\mathbf{j} + (e^{3t} + 3)\mathbf{k}$$

at the point $(2, 3, 4)$.

- $\vec{h}(t) = (2,3,4) \iff \begin{cases} 2 = e^t + 1 \\ 3 = e^{2t} + 2 \\ 4 = e^{3t} + 3 \end{cases}$

(1st eqn) $\Rightarrow$ $1 = e^t \Rightarrow \ln(1) = t = 0.$

$\vec{h}(0) = (2, 3, 4)$ (Since $e^0 = 1$).

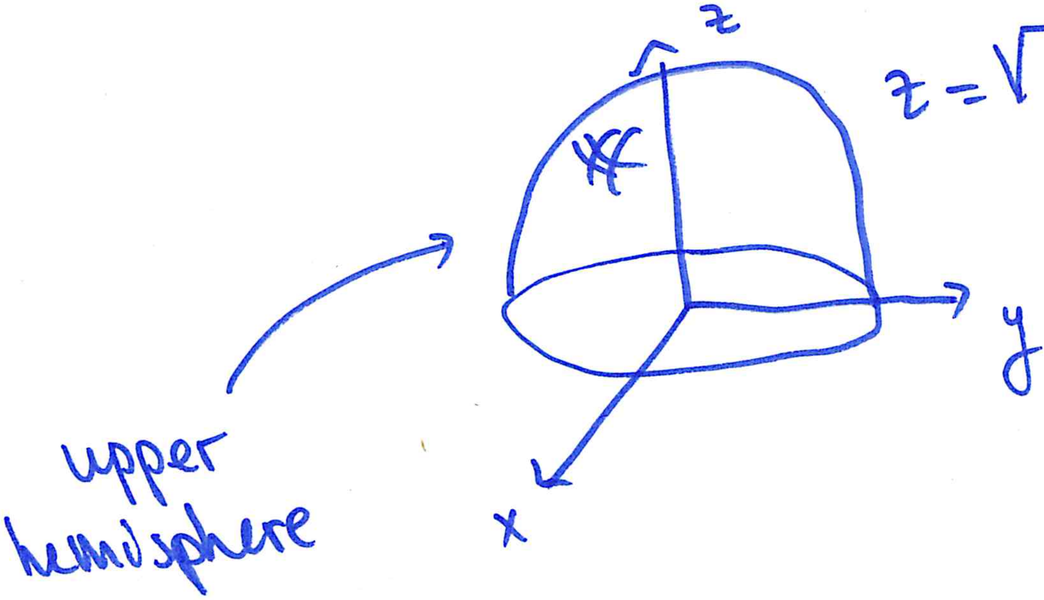
- $\vec{h}'(t) = (e^t, 2e^{2t}, 3e^{3t})$

$\vec{h}'(0) = (1, 2, 3)$ = direction vector.

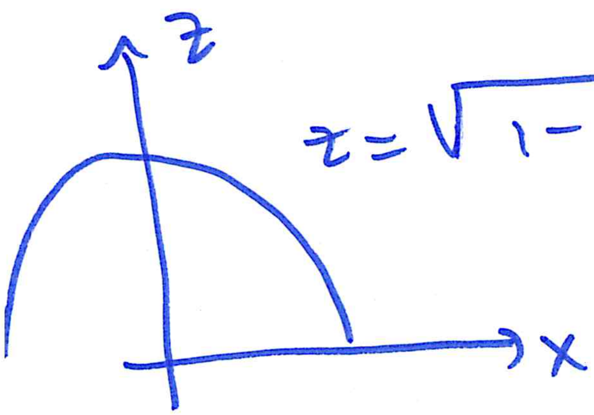
Putting these together,

$\vec{l}(\tau) = (2,3,4) + \tau(1,2,3).$ //

5. (10 points) Graph the function $f(x, y) = \sqrt{1 - x^2 - y^2}$ for $0 \leq x^2 + y^2 \leq 1$. (Remember that including more information, like traces and labels makes your graph easier to understand.)

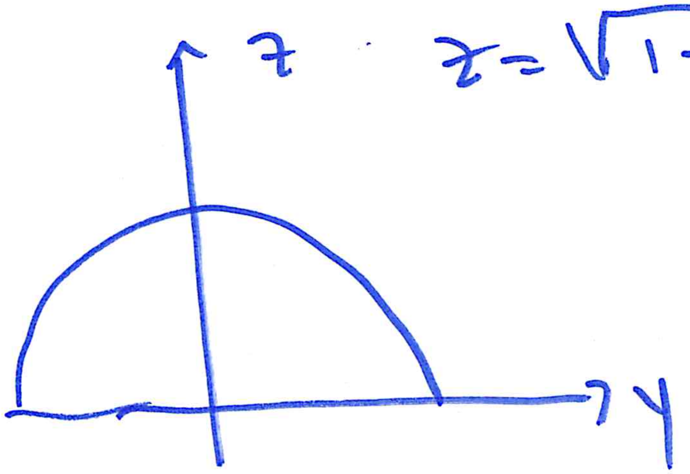


$z = \sqrt{1 - x^2 - y^2}$



$z = \sqrt{1 - x^2}$

xz-trace



$z = \sqrt{1 - y^2}$

yz-trace

6. (20 points) Consider the lines parametrized by

$$\begin{aligned} \alpha(t) &= (t, 1-t, 0), \\ \beta(t) &= (t/2, t/2, 1-t) \end{aligned}$$

(a) Find where the lines parametrized by $\alpha(t)$ and $\beta(t)$ intersect.

(b) Determine an equation for the plane containing the lines parametrized by $\alpha(t)$ and $\beta(t)$.

(You may want to do part (b) on the back to give yourself some space.)

(a) $\vec{\alpha}(t) = \vec{\beta}(\tau)$

$$\Leftrightarrow \begin{cases} t = \tau/2, \\ 1-t = \tau/2, \\ 0 = 1-\tau, \end{cases} \Rightarrow \tau = 1.$$

$\vec{\beta}(1) = (1/2, 1/2, 0)$, you can see $t = 1/2$ works for 2nd component.

check: $\vec{\alpha}(1/2) = (1/2, 1-1/2, 0) = (1/2, 1/2, 0)$.

(point of intersection) $= (1/2, 1/2, 0)$.

(b) Need to find the normal vector.

$\vec{v}_1 = (1, -1, 0)$

$\vec{v}_2 = (\frac{1}{2}, \frac{1}{2}, -1)$

$$\vec{V}_1 \times \vec{V}_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 1/2 & 1/2 & -1 \end{vmatrix}$$

$$= i \begin{vmatrix} -1 & 0 \\ 1/2 & -1 \end{vmatrix} - j \begin{vmatrix} 1 & 0 \\ 1/2 & -1 \end{vmatrix} + k \begin{vmatrix} 1 & -1 \\ 1/2 & 1/2 \end{vmatrix}$$

$$= i(1) - j(-1) + k(1/2 + 1/2)$$

$$= i + j + k = \vec{n}.$$

$$\therefore \quad \vec{n} \cdot (\vec{r} - \vec{r}_0)$$

$$= (1,1,1) \cdot (x - 1/2, y - 1/2, z)$$

$$= x - 1/2 + y - 1/2 + z$$

$\Rightarrow$ Equation of plane:

$$x + y + z - 1 = 0.$$

7. (20 points) Assume a and b are non-zero constants.

(a) Find a parametrization of a line in $\mathbb{R}^3$ passing through the points $(0,0,1)$ and $(a,b,0)$.

$$\vec{\alpha}(t) = (0,0,1) + \left((a,b,0) - (0,0,1)\right)t$$
$$= (0,0,1) + (a,b,-1)t$$
$$= (at, bt, 1-t).$$

(b) Where does this line intersect the plane $x = 1$?

$$at = 1 \Rightarrow t = 1/a.$$

$$\vec{\alpha}(1/a) = (a(1/a), b(1/a), 1 - 1/a)$$
$$= (1, b/a, 1 - 1/a). //$$

8. (10 points)

(a) State the definition of the unit tangent, unit normal and unit binormal vectors of a curve parametrized by $\mathbf{r}(t)$.

$$\vec{T}(t) = \vec{r}'(t) / |\vec{r}'(t)|$$

$$\vec{N}(t) = \vec{T}'(t) / |\vec{T}'(t)|$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t).$$

(b) Show that unit tangent and unit normal vectors of $\mathbf{r}(t)$ are perpendicular. (Hint: take the derivative of both sides of $|\mathbf{T}(t)|^2 = 1$. Use the dot product rule for the LHS.)

$$\frac{d}{dt}\left[\vec{T}(t) \cdot \vec{T}(t)\right] = 0$$

$$\Rightarrow 2\underbrace{\vec{T}'(t)}_{\text{same direction as unit normal vector.}} \cdot \vec{T}(t) = 0$$